An Investigation into the Extraction of Melodic and Harmonic Features from Digital Audio

by

Daniel Zachariah Franks

Thesis presented in fulfilment of the requirements for the degree of Magister Philosophiae of Music Technology in the Faculty of Arts and Social Sciences at the University of Stellenbosch

Department of Music
University of Stellenbosch
Private Bag X1, 7602 Matieland, South Africa

Supervisor: Mr. Gerhard Roux

March 2017

Stellenbosch University https://scholar.sun.ac.za

Declaration

By submitting this thesis electronically, I declare that the entirety of the work contained therein is my own, original work, that I am the sole author thereof (save to the extent explicitly otherwise stated), that reproduction and publication thereof by Stellenbosch University will not infringe any third party rights and that I have not previously in its entirety or in part submitted it for obtaining any qualification.

Date: ....March 2017.............

Copyright © 2017 Stellenbosch University
All rights reserved.

Stellenbosch University https://scholar.sun.ac.za

Abstract

An Investigation into the Extraction of Melodic and Harmonic Features from Digital Audio

D.Z. Franks

Department of Music
University of Stellenbosch
Private Bag X1, 7602 Matieland, South Africa

Thesis: MPhil Music Technology

March 2017

Approaches towards musical pitch analysis by software are presented with unique interpretative challenges in the inherent complexity of presenting results that are not only adequate for scientific researchers but also of relevance to musical practitioners. The cognitive representation of musical pitch arrangement is tied to societal conventions, canonical practices and their listeners expectations. These conventions are by no means universal. Different traditions of music may make very different uses of pitch space, and have very different associated ideas of 'tonalities', and intervals. It is considered in this research that a complex mathematical model of pitch space may flesh out a suitably unbiased model of musical sensation with which one may describe the many discrepancies that exist between, and within, differing sociologically defined canons of musical pitch theory. This research describes such a practical approach towards such a suitably complex methodology of pitch data interpretation and proposes a music information retrieval system based upon these findings.

Stellenbosch University https://scholar.sun.ac.za

Uittreksel

’Õn Ondersoek na die myn van melodiese en harmoniese eienskappe vanuit digitale opnames

D.Z. Franks

Departement Musiek
Universiteit van Stellenbosch
Privaatsak X1, Matieland, 7602, Suid-Afrika

Tesis: MPhil Musiektegnologie

Maart 2017

Benaderings tot toonhoogte-analise in musiek deur middel van sagteware staar unieke uitdagings in die gesig ten opsigte van die lewering van resultate wat geskik is vir beide navorsing en die praktyk. Die kognitiewe voorstelling van musikale toonhoogte berus op maatskaplike konvensie, kanonieke praktyke en die verwagtings van luisteraars. Verskillende tradisies van musiek wend die toonhoogtespektrum op verskillende wyses aan en het uiteenlopende idees omtrent ‘tonaliteit’ en intervalle. Hierdie navorsing wend ’n onbevooroordeelde toonhoogtepersepsiemodel aan om die verskille tussen die verskeie sosiologiese interpretasies van toonhoogte uit te lig. ’n Metodologie om toonhoogtedata te interpreteer word in hierdie navorsing geïmplementeer en ’n inligtingherwinningstelsel is op grond daarvan ontwikkel.

Stellenbosch University https://scholar.sun.ac.za

Acknowledgements

I would like to express my sincere gratitude to my supervisor, Mr. Gerhard Roux, for his persistent efforts throughout this research. His assistance and understanding has played no small part in making this research possible. My greatest thanks go to my family, who have put up with me throughout the writing of this research, and without whom I would not be able to pursue this work. To my friends and colleagues in music, I wish to express my thanks for the many joyful hours spent in discussion.

Stellenbosch University https://scholar.sun.ac.za

Dedications

To god, the muses, and music. To my fellow musicians and researchers.

Stellenbosch University https://scholar.sun.ac.za

Contents

Stellenbosch University https://scholar.sun.ac.za

Stellenbosch University https://scholar.sun.ac.za

Stellenbosch University https://scholar.sun.ac.za

List of Figures

Stellenbosch University https://scholar.sun.ac.za

Stellenbosch University https://scholar.sun.ac.za

List of Tables

Stellenbosch University https://scholar.sun.ac.za

Nomenclature

Common Musical Acronyms

n-TET = n - Tone Equal Temperament .... eg. 12-TET, 19-TET.

n-EDO = n - Equal Divisions of an Octave .... eg. 31-EDO, 53-EDO.

Glossary of Musical Terms

Pitch = "A pitch is the perceived fundamental frequency of a tone." (Lots & Stone, 2008:6)
"The pitch number of a note is commonly called the pitch of the note. By a convenient abbreviation we often write a' 440, meaning the note a' having the pitch number 440; or say that the pitch of a' is 440 vib. that is, 440 double vibrations in a second. [...] The pitch of a musical instrument is the pitch of the note by which it is tuned. But as pitch is properly a sensation, it is necessary here to distinguish from this sensation the pitch number or frequency of vibration by which it is measured. The larger the pitch number, the higher or sharper the pitch is said to be. The lower the pitch number the deeper or flatter the pitch. These are all metaphorical expressions which must be taken strictly in this sense." (Helmholtz, 1895:11)

Frequency = "[...] frequency, [...] much used by acousticians, properly represents the number of times that any periodically recurrinq event happens in one second of time, and, applied to double vibrations, it means the same as pitch number." (Helmholtz, 1895:11)"

Octave = "... a musical tone which is an Octave higher than another, makes exactly twice as many vibrations in a given time as the latter." (Helmholtz, 1895:13)

Stellenbosch University https://scholar.sun.ac.za

Tuning = "A tuning system is defined here to be a collection of precisely tuned musical intervals. There are many ways in which the intervals may be chosen: A "boutique" tuning system might have all of its intervals chosen arbitrarily, and another tuning system might be generated by a predefined mathematical procedure. The regular tunings form one class of tuning systems in which all of the intervals are generated multiplicatively from a finite number of generating intervals (or generators). Such tuning systems ensure that every given note has the same set of intervals above and below it as every other note in the system; this means that regular tuning systems are inherently transpositionally invariant. An example regular tuning system is 3-limit JI (also known as Pythagorean tuning), which has two generators $G_1 = 2$ and $G_2 = 3$, and consists of all products of the form $G_i \; G_j = 2_i$ x 3_j,where i and j are integers. Thus the intervals of 3-limit JI can all be found in a series of stacked just perfect fifths, allowing for octave equivalence. In general, a regular tuning is characterized by n generators G_1 to G_n and consists of all intervals ${G_2}^{i1} \; {G_2}^{i2} \ldots {G_n}^{in}$, where the $i_1 \; i_2, \; . \; . \; . \; , \; i_n$ are integer-valued exponents. Altering the tuning of a generator affects the tuning of the system in a predictable way. For example, the perfect fifth in 3-limit JI is $G_1 \, ^{-1} \; G_2$ (i.e., 2^{-1} x 3)." (Milne *et al.*, 2007:20)

Temperament = "[...] temperament may be defined to consist in slightly altering the perfect ratios of the pitch of the constituents of a chord, for the purpose of increasing the number of relations between chords, and facilitating musical performance and composition by the reduction of the number of tones required for harmonious combination" (Ellis, 1863*b*:404)

Ornamentation = Originating in improvised performance technique, "[t]he more or less stereotyped melodic figures that [are] substituted for or added to the original notes of the melody are known as ornaments" (Apel, 1983:629).

Nonharmonic = Also nonmelodic."In harmonic analysis, [a] generic term for tones that are foreign to the harmony of the moment and occur as melodic ornamentations in one of the parts." (Apel, 1983:576)

Melody = Broadly defined, a melody is a succession of musical sounds. (Apel, 1983:517)

Harmony = "The chordal (or vertical) structure" of musical sounds. (Apel, 1983:371)

Stellenbosch University https://scholar.sun.ac.za

Modality = Modality is a historical musical system of pitch relations. Common amongst classical musical systems prior to the seventeenth century, the organization of mosal pitch distributions is based upon melodic principles rather than chordal Dahlhaus (1990), and may not necessarily be reduced to a strongly defined tonic (Milne, 2013:7).

Atonality = "Atonality refers to those systems of music developed in the twentieth century (notably serialism), which deliberately avoid structures that generate a tonic." (Milne, 2013:7)

Pure Tone = "A *Pure tone* is a single frequency tone with no harmonic content (no overtones). This corresponds to a sine wave. It is characterized by the frequencyÑthe number of cycles per second and the amplitude of the cycles." (Lots *&* Stone, 2008:6)

Interval = "In music theory, the term interval describes the difference in pitch between the fundamental frequencies of two notes." (Lots *&* Stone, 2008:6)

JND = Just Noticeable Difference. It has been usefully established that the just-noticeable difference for consecutive pitches, and limit of discrimination for simultaneous pitches, is dependent upon both frequency and intensity and ranges from 220 cents for very quiet ($5dB$) very low ($31hz$) tones, a large margin, to as little as 3 cents for tones around $1000hz$ at 30 dB. (Benson, 2007:15-16).

Regular Systems = "*Regular systems* are such that all their notes can be arranged in a continuous series of equal fifths.[1] (Bosanquet, 1874:391)

Cyclical Systems = *Regular cyclical systems* are not only regular, but return into the same pitch after a certain number of fifths. Every such system divides the octave into a certain number of equal intervals. (Bosanquet, 1874:391)

Error = *Error* is deviation from a perfect interval. (Bosanquet, 1874:391)

Departure = *Departure* is deviation from an E. T. interval." (Bosanquet, 1874:391)

[1] "The importance of regular systems arises from the symmetry of the scales which they form." (Bosanquet, 1874:392)

Stellenbosch University https://scholar.sun.ac.za

Invariance = "... patterns which remain the same in different contexts." (McClain, 1978:7)

Integral = Ratios involving the use of simple whole numbers (i.e. numbers from 1- 9), eg. $^3/_2$, and $^9/_8$.

Enharmonic = Ratios involving the use of polynomials such as the number 10, eg. $^{36}/_{35}$, and $^{28}/_{27}$.

Gamut = A range of sounded pitches. Used in this research to describe the notes ranging within an octave, from unity $^1/_1$ to the octave $^2/_1$, usually reduced theoretically by octave equivalences from perceived pitch usage.

Mathematical Notation

$\mathbb{N}$ = Natural Numbers, the set of whole numbers.

$\mathbb{R}$ = Real Numbers, including natural numbers, rational numbers and irrational numbers.

$\mathbb{Q}$ = Rational Numbers, ratios of whole number relations.

$\mathbb{A}$ = Algebraic Numbers, including irrational square roots, square and cube numbers.

Arithmetic mean = Arithmetic means are points arranged in such a way that they construct equally sized, and unequal proportioned, parts. (Barker, 1989:42).

> "an intermediate value between two extremes; there is always a larger ratio between the smaller numbers. (For example, 3 is the arithmetic mean between 2 and 4, but the ratio 2:3 is a musical fifth while the ratio 3:4 is a musical fourth.)
> formula: $M^a = \frac{A+B}{2}$" (McClain, 1978:xvi)

Geometric mean = Geometric means are points arranged in equal proportions of unequally sized parts. (Barker, 1989:42).

> "that intermediate value which divides an interval proportionally (into two intervals with the same ratio), as for instance 2 is the geometric mean between 1 and 4 and 3 is the geometric

Stellenbosch University https://scholar.sun.ac.za

> mean between 1 and 9. In equal-temperament the semitone is geometric mean within the wholetone, the wholetone within the major third, the major third within the augmented fifth, etc., but in ancient tunings it was present in the scale only under exceptional circumstances." (McClain, 1978:xviI)

Harmonic mean = Harmonic means are points arranged by progressively unequal proportions and similarly unequal parts (Barker, 1989:42).

> "the "sub-contrary" of the arithmetic mean, with the larger interval between the larger numbers. For instance, 9 is the arithmetic mean in the octave double 6:12 (dividing it into a fifth of ratio 2:3 and a fourth of 3:4) while 8 is harmonic mean with the fifth of 2 :3 at 8 :12 and the fourth of 3:4 at 6:8." (McClain, 1978:xviI)

Stellenbosch University https://scholar.sun.ac.za

Chapter 1

Introduction to Pitch Structure, Analysis and History

In such a field, where necessity is paramount and nothing is arbitrary, science is rightfully called upon to establish constant laws of phenomena, and to demonstrate strictly a strict connection between cause and effect. As there is nothing arbitrary in the phenomena embraced by the theory, so also nothing arbitrary can be admitted into the laws which regulate the phenomena, or into the explanations given for their occurrence. As long as anything arbitrary remains in these laws and explanations, it is the duty of science (a duty which it is generally able to discharge) to exclude it, by continuing the investigations.

– Helmholtz (1895:234)

1.1 Introduction

THE nature of the scientific representation of sound and musical tone involves many complicated challenges. We may treat many of these complications as arising from two simple, inextricable questions. Quantitatively, by what measurable sensations do sounds reveal themselves to perception as musical? Qualitatively, what is the meaning of these sensible measurements by which sound may encode itself as music?

We may answer the first question by systematically describing spatial analogies of height, depth and linear progression, determining between positions of phenomena and their intervals, and from these determinations outlining ab-

Stellenbosch University https://scholar.sun.ac.za

stracted spatial structures[1] that may usefully describe pitch metrics and scales. Such metrically mapped pitch spaces[2] are useful constructs for describing the perceived relations of sound sensations that correspond to musical tone. Applying mathematical methods to musical pitch space allows for great varieties of temperaments and tuning to be represented with exacting multi-dimensional detail using a variety of inter-related metric scalings[3]. The second question is more difficult. Qualitatively, these phenomena show themselves to be variously defined by various schools of musical theory and national taste (Helmholtz, 1895:234).

Quantitative distinctions are subject to a great variety of qualitative systematisation in musical pitch theories. Representations of such musical theories and their analyses, whether mapped by a musical stave or a graphed dimensional axis, while implicitly accepting various mathematical complexities of spatial constructs, allow for only such resolutions of pitch data as are capable of formal statement by their musical pitch theories.

While such methods may be capable of general degrees of pitch analysis[4], the generalised schemes of these notations does not commonly negotiate certain complexities regarding pitch usage, such as those involved in the classical oriental methods and occidental folk practices of ornamentation[5]. Included in this term are the many varieties of non-melodic successions, deflections and pitch alterations and adjustments, together with the melodic grace functions such as trills, appogiatura, acciacatura, mordents, glissando, portamento and other pitch slurs. A sufficiently complex theory, with a suitably resolute refined pitch structure, capable of defining such exacting pitch deviations may offer a well-fitting representation for modelling exact pitch practices.

It has been shown that the ear organises sounds in terms of pitch according

[1] That is, a structure that is proposed to have spatial features and yet is insubstantial, a spatial construct existing only in the abstract, such as that proposed by the pitch-space analogy.

[2] As abstracted spatial constructs defining a pitch scale which is measurably mapped to a metric scaling. Just such a construct is usefully furnished by the mathematically defined monochord divisions of classical Greek harmonic theory.

[3] A metric scaling is quite literally a mathematically precise scaling of pitch space, usually related in terms of physical model, i.e.: string length or air column represented as fractional rational relations of whole numbered harmonic relations or approximated irrational logarithmic functions.

[4] Notably, such as that regarding classical Western theory and its notation of pitch by stave and height, or the Indian classical techniques of pitch representation by syllables and various alterations, both of whom commonly acknowledge a resolution of twelve pitch identities to the octave. The methods of classical Greek alphabetic representation, which noticeably influenced modern western classical representations of pitch identities and it's common twelve tone analysis, were of a much higher order of mathematical pitch identification than the aforementioned contemporaneously popular pedagogic theories.

[5] The many varieties of these ornaments are based inherently upon mathematical divisions.

Stellenbosch University https://scholar.sun.ac.za

to exacting mathematical methods[6], and allows for very fine pitch distinctions to be made[7]. These differences, obscured by ambivalent approaches to enharmonic equivalencies, as encapsulated in generalised 12-tone pitch class theory, are a principle issue for practically indicating interval differences. These interval differences being essential for differentiation amongst existing tunings and temperaments in terms of their representations of pitch classes, and also for exactingly representing theories of grace ornaments.

The practical usage suggested by these *non-melodic successions*[8], as drawn from the translated writings of Aristoxenus contained in Aristoxenus & Macran (1902) and Barker (1989), suggest a theoretical inclusion of enharmonic discrepancies of pitch together with other differences in the exact pitch relations of harmonic and melodic interval classes[9].

The development of complex and mathematically rigorous models of musical pitch structure have been the topic of much musical research. Investigations of scientific researchers throughout the 19th century, and the years preceding, laid down many of the scientific principles of acoustics.[10], These early investigations in search of primary acoustic principles, in many cases, concern the description of music in terms of mathematical relations describing spatial analogies.

Different theoretical canons of musical systems have undergone various degrees of development, and while many of these may be similar, they bear remarkable differences in many cases[11]. These differences underpin widely variable relations of theoretical models of music and practical application, especially concerning rhythmic awareness [12] and intonation[13].

The divergent histories of the musical arts and the acoustic sciences have obscured many of the links between canonic theories and social practices. In very few ways does the representation of sound by scientific units resemble the interpretation of musical practices from notation, direction, and tutelage.

---

[6] (Helmholtz, 1895:49-65)

[7] The distinctions that are theoretically capable by the hairs of the cochlea are in the region of 2 cents, or $1/600^{th}$ of an octave, however the potential range of these distinctions is dependent to some degree on the limitations of the basilar membrane(Helmholtz, 1895:49-65, 406-411). Also see the entry on Just Noticeable Difference (JND) in the glossary. For further detail see Pierce (1983), Benson (2007:15-16), and Backus & Baskerville (1977).

[8] See glossary.

[9] This inclusive approach to pitch deviation is further explicated in subsection 3.2.2.

[10] Some of these developments are described in chapter 3

[11] For examples of these various developments see Cho (2003), Barker (1989), Bosanquet (1877), Erlich (1998), West (1992), and also the descriptions in chapters 2 and 3.

[12] Extemporalisation, capacity for syncopation, and systems of divisions and additions of rhythm and metric.

[13] Tuning varieties, regular or irregularly ordered temperament schemes, and approaches to ornamentation, melodic succession and harmonic arrangement.

Stellenbosch University https://scholar.sun.ac.za

They may share the common features of spatial analogy[14], the use of similar common variables[15], and the representation of relative tensions of variables[16], and yet they differ substantially in their presentation of these common considerations. Detailed scientific representations of music, and musical notation for that matter, bear very little immediate resemblance to the sensations produced by the phenomena of musical tone, although, given years of technical training, one may learn to interpret such notations and scientific representations. We may seek the reasons for this incongruence of the art to the science in history, considering how the arts of music have tended in general towards making use of acoustic researches only in order to ease the complications of musical theory. This tendency to simplification of musical theory, while useful for initial instruction in music, by taking instrumental advantage of advances in acoustic sciences[17] comes to deny the complications of musical theory and its history. Knowledge regarding intonation and tuning, being inherently useful for dividing an unfretted string or for placing moveable frets, becomes unnecessary when one has ready access to fixed-pitch sounds, such as constitute fixed-fret string divisions, a modern piano keyboard, accordion buttons, and MIDI[18] triggers. Many theoretical vaguenesses and ambiguities of terminology are introduced by ignorance of such complex perspectives on pitch relations and intonation[19]. Generalisations regarding pitch persist in many pedagogic models of modern music theory, becoming in many cases resistant to any more precise definition[20].

[14] i.e. Pitch and pitch relations may be analogously represented as heights and distances. Progressions by sequencing. These generalised notions are common amongst notation systems, theoretical relations, and other colloquial descriptions and traditions of musical instruction and transmission. For further exposition of these spatial analogies and their correlations, see subsection 1.1.2 below, section 2.2.3, and also Pesic (2013); Brower (2008).

[15] Such as the common relation underlying sounds in their incarnations as notes, pitches, and frequencies, and events having duration and time scale.

[16] In terms of the representation of interval and pitch relations as phenomena having relative tension and attraction.

[17] Such as may allow for mechanistic fixed pitch instruments to be manufactured, and for theories of performance practice that are divorced from such technical concerns as tuning and intonation to come to dominance. See Ellis (1863*b*,*a*) for further discussion of these and other factors of fixed pitch practices.

[18] Musical Instrument Digital Interface, a protocol for the control of synthesizers and other digital instrument, was developed in 1983 at a conference held by various musical instrument manufacturers. This protocol, and its various hardware design specifications, have contributed considerably to compatibility between electronic devices produced by different instrument manufacturers. Association *et al.* (1983) See Loy (1985) for further detail regarding these developments.

[19] See chapters 2 and 3 for further problematisation concerning pitch relations and pitch identities.

[20] See Duffin (2008), Busoni (1911:4), and Partch (1949).

Stellenbosch University https://scholar.sun.ac.za

The early investigations of 19th century science into the nature of musical sound, and the sensations at its cause, were restricted to mechanical and philosophical investigations until successful attempts at electrical signal analysis proved their utility (Helmholtz, 1895:20, 372).

The basic principles of these considerations are still the same concerns with which the earliest acoustic researchers dealt, the development of reliably analogous constructs with which to represent and reproduce musical sound relations. Elaborations of the desired analogous construct are of necessity tempered by whatever ends they are to serve and the limitations underlying the probative and reproductive methods of their investigation.

The 20th century development of modern computing technology has expanded the nature of these analogies further, from the simple electrical analogy modern computing science has elucidated many information based, *datative*, electrical analogies.

An analogous current of voltage representing a musical signal with many parameters may be reproduced by a computer data bank as a digital stream of information. This may be coded and decoded many times, read and stored in multiple formats, rendered as graphed data figures, or reproduced as sound.

In terms of efficient tools for scientific signal analysis, modern computing methods have expanded upon the methods of the early acoustic investigators and their mechanical methods of relativistic determinations of sound relations[21].

Representations of data regarding sound and musical arrangement may be rendered intricately, and quickly, by modern computer analysis, and to a higher degree of resolution than a human observer is capable of.

Notwithstanding this, mechanical representation remains a most direct and useful analogy to practical musical instructions, as is rendered by a study of monochord divisions[22], or the harmonics of various pipes, membranes and any other sound producing massed substances.

1.1.1 History

> Long before anything was known of pitch numbers, or the means of counting them, Pythagoras had discovered that if a string be divided into two parts by a bridge, in such a way as to give two consonant musical tones when struck, the lengths of these parts must be in the ratio of these whole numbers. If the bridge is so placed that $2/3$ of the string lie to the right, and $1/3$ on the left, so that the two lengths are in the ratio of 2 : 1, they produce the interval of an Octave, the greater length giving the deeper tone. Placing the bridge so that $3/5$ of the string lie on the right and $2/5$

[21] The principles of sympathetic resonance (Helmholtz, 1895:36-49).

[22] See Adkins (1963) and Creese (2010).

Stellenbosch University https://scholar.sun.ac.za

> on the left, the ratio of the two lengths is 3 : 2, and the interval is a Fifth. (Helmholtz, 1895:14)

The prevailing scientific view of pitch structure is based upon the manifold structures of mathematics; arithmetic, geometry, and algebra. This mathematics is used in the devising, manufacture and tuning of instruments, however the theoretical implications of this is often taken for granted by performing musicians performing upon instruments with fixed pitches based upon the keyboard manual, fixed-frets, and 12 tone equal temperament (12-TET) theory.

Modern musical theory, in general, regards twelve pitch classes as being sufficient for the purposes of musical education and the analysis of melodic and harmonic motion. For practical purposes, these twelve pitch classes may permit of various acceptable tuning schemes[23]. The variety of these tuning conventions, and the resulting variety of the mathematically explicit definitions of pitch that they entail, have been of great interest to researchers interested in the analysis of musical pitch structures. Such tuning schemes and intonation are complex issues involving the histories of instruments, luthiers, mathematics and musical practices.

The cosmologically bound conception of music and numbers forged by classical Greek scholarship deserves some brief mention. Denuded of some of its more poetic and uncertain aspects, the quantitative methods of Pythagoreanism, elucidate a scientific method of acoustic research into musical tone and the measurement of pitch space.

> These measurements had been executed with great precision by the Greek musicians, and had given rise to a system of tones, contrived with considerable art. (Helmholtz, 1895:14)

The contrivance of this musical development owed much to the unique mechanism of string instruments, and in particular to the monochord, "a peculiar instrument [...] consisting of a sounding board and box on which a single string was stretched with a scale below, so as to set the bridge correctly." (Helmholtz, 1895:15)

The demonstrative powers of the monochord, though exhibiting the law of inverse proportions[24], requires for this determination some method of observing vibration numbers of the periodic sounds produced by relative string lengths. "It was not till much later that, through the investigations of Galileo (1638),

[23] Such as the Pythagorean method of tuning fifths, the many widely varying standards of Just-Intonation, logarithmically tempered Equal Temperaments, and other varying schemes of unequal temperaments. These are treated further in chapters 2 and 3.

[24] The frequency number of a stretched string varies inversely to its length. i.e. Length $\times$ Pitch Number $= 1$, therefore Pitch $= {}^{1}/_{Length}$, and also Length $= {}^{1}/_{Pitch}$, and further the frequency of vibration numbers is in direct proportion to the square root of the force of tension(Scherchen, 1950:18).

Stellenbosch University https://scholar.sun.ac.za

Newton, Euler (1729), and Daniel Bernouilli (1771), the law governing the motions of strings became known, and it was thus found that the simple ratios of the lengths of the strings existed also for the pitch numbers of the tones they produced, and that they consequently belonged to the musical intervals of the tones of all instruments, and were not confined to the lengths of strings through which the law had been first discovered." (Helmholtz, 1895:15)

From this investigation of the physical causes of sensation arose the need for special consideration of how this sensation manifests to the perception. Investigations into the perception of musical sound tend to locate their relations within spatial frameworks describing and governing the pitch distributions of musical relations.

1.1.2 Pitch Structure

The information afforded to sensation in regards to the musical perception of sounds reveal many useful notions. The basic difference between musical sounds and noise consists in the nature of their movements. Passing through the atmospheric medium sound waves may be regular or irregular in their motion. The regular motion of air produced by the equally regular motion of a sonourous body gives rise to sounds we term musical tones. Irregular motions however sound as mere noise (Helmholtz, 1895:8).

> Those regular motions which produce musical tones have been exactly investigated by physicists. They are oscillations, vibrations, or swings, that is, up and down, or to and fro motions of sonorous bodies, and it is necessary that these oscillations should be regularly periodic. By a periodic motion we mean one which constantly returns to the same condition after exactly equal intervals of time. The length of the equal intervals of time between one state of the motion and its next exact repetition, we call the length of the oscillation, vibration, or swing, or the period of the motion. (Helmholtz, 1895:8)

The sensation of musical tone is differentiated by perception from that of sounding noise. Regular periodicity describes relations which sound as musical tones. Theorising of musical pitch is concerned with describing such states and changes as may occur amongst and within these periodic relations.

> We are acquainted with three points of difference in musical tones, confining our attention in the first place to such tones as are isolatedly produced by our usual musical instruments, and excluding the simultaneous sounding of the tones of different instruments. (Helmholtz, 1895:10)

Stellenbosch University https://scholar.sun.ac.za

Isolated musical tones may be distinguished by three characteristics belonging to periodic musical sounds, their force, pitch and quality (Helmholtz, 1895:10).

Force , as a measure of loudness, relates to the amplitude of oscillation of the periodic motion under consideration[25].

Pitch is a proportionate measure regarding the time scale of periods of vibrations[26].

Quality is a measure of the integral proportions of each periodic motion and defines timbral aspects of tonal phenomena[27].

Distinguishing a sound's force, pitch and quality are necessary distinctions for the purposes of musical and scientific analysis. These distinguishing characteristics provide quantitative relations by which we may form a mathematically related observation. Regarding pitch thus, Helmholtz outlined the scientific units of pitch number and frequency.

> We are accustomed to take a second as the unit of time, and shall consequently mean by the pitch number [or frequency] of a tone, the number of vibrations which the particles of a sounding body perform in one second of time.[28] (Helmholtz, 1895:11)

A direct relation of pitch to sound may be discreetly rendered by electrical analogy. A sound may be represented by an electrical analogue, by a microphone or other pickup, as a signal. Such an electrical signal represents the force of the sound by the amplitude of its voltage strength, the pitch of the sound by the frequency of the electrical signals waveform, and the timbral quality of the sound in the compound form of the wave.

The musical relations that determine key centre and interval generators[29] when applied to signal analysis may be used to plot frequency relations mathematically against a fundamental frequency, or f_0. This is determined in prac-

[25]"[...] loudness must depend on this amplitude, and none other of the properties of sound do so." (Helmholtz, 1895:10)

[26]"Pitch depends solely on the length of time in which each single vibration is executed, or, which comes to the same thing, on the number of vibrations completed in a given time." (Helmholtz, 1895:11) "Pitch is the perception of how high or low a musical note sounds, which can be considered as a frequency which corresponds closely to the fundamental frequency or main repetition rate in the signal." (McLeod & Wyvill, 2005:1)

[27]"By the quality of a tone we mean that peculiarity which distinguishes the musical tone of a violin from that of a flute or that of a clarinet, or that of the human voice, when all these instruments produce the same note at the same pitch." (Helmholtz, 1895:10)

[28]Pitch number was called 'vibrational number' in the first edition of Ellis' translation of Helmholtz (1895:11).

[29]Dealt with further in chapters 2 and 3.

Stellenbosch University https://scholar.sun.ac.za

tice by correlating the number of pitch occurrences and their relative amplitudes, with the result that the maximum values indicate this fundamental frequency.

> Once f_0 is known, a full harmonic analysis of the sound becomes possible [...] and we can display many other aspects of a sound that are useful to a musician. (McLeod & Wyvill, 2005:1)

The determination of fundamental frequency (f_0) is analogous to the musical discrimination of pitch, and from this we may construct a musically useful analysis of pitch characteristics with which to base an analysis of the signal.

Constructing a useful analysis, in terms of musical pedagogy and the theoretical and practical understandings of performing musicians, requires further correlation between musical description and the mathematical relations of the signal. The wide variation amongst schools of theory regarding musical pitch structure require some explanation.

1.1.3 Musical Analysis

The phenomena of tone (the perception of sound, and of successive and simultaneous pitch relations) are not capable of easy separation. In the art of geometry a point is a thing with no parts, capable of dividing and relating things that do have parts[30]. Such is the idea of a musical note, as a notion that relates various perspectives of consideration of points, sounds produced by lengths of strings, volumes of air. The terminology used to describe such relations constitute a wide variety of practical nomenclature.

Musical notations such as western notation and tablature provide useful generalised nomenclature, though without some extension[31] these do not fully elucidate the specific nature of the practical realisation that is sought in this research.

By the term 'octave' is generally understood some notion of the 'same note' being available in a higher / lower register, though it still remains to express what makes these notes similar. That a string length may be doubled or halved to provide octave relations, and divided into thirds to obtain 'fifth' relations, is common knowledge amongst string playing musicians. Less commonly known are the mathematical interplay of five-limit intervals upon a monochord[32]. Without a mathematically grounded topology of pitch space there can be none but 'fuzzy' distinctions of pitch when discussing such topics as the varieties of the various tones and minor and major semitones. Without such definitive mathematics it is very difficult to understand the natural contexts of the dieses

[30] See Aristides Quintilanus in Barker (1989:436).

[31] Such as the conventions of Helmholtz-Ellis notation described in subsection 3.1.4.

[32] See sections 2.1.3 and 3.3.2.

Stellenbosch University https://scholar.sun.ac.za

and commas, fractional semitones and tones, and there can be no distinction of enharmonic relations.

Helmholtz observed from his experiments with sirens and studies of other sounding bodies that integer proportions result in simple musical relations. A proportion of 1 to 2 results in an octave, a proportion of 2 to 3 gives a fifth, a proportion of 3 to 4 gives a fourth, a proportion of 4 to 5 gives a major third, and a proportion of 5 to 6 gives a minor third (Helmholtz, 1895:14).

Octave transposition of these simple pitch relations furnish inversions of their relations [33].

> Thus a Fourth is an inverted Fifth, a minor Sixth an inverted major Third, and a major Sixth an inverted minor Third. (Helmholtz, 1895:14)

These descriptions define, and are defined by, points and relations agreeing with the theory of monochord division, and the schools of mathematical thought which permeate historic conceptions of music.[34]

Throughout history musicians have made use of various conventions in the performance of their different musics. These are usually modelled simply by the use of points and their relations. These constitute what western theory calls notes[35], intervals[36], scales[37], registers[38], octaves[39] and many other related terms whose full description is beyond the scope of this research.

---

[33]"When the fundamental tone of a given interval is taken an Octave higher, the interval is said to be inverted. The corresponding ratios of the pitch numbers are consequently obtained by doubling the smaller number in the original interval. [...] From 2 : 3 the Fifth, we thus have 3 : 4 the Fourth. From 4 : 5 the major Third , we thus have 5 : 8 the minor Sixth. From 5 : 6 the minor Third, we thus have 6 : 10 = 3 : 5 the major Sixth. These are all the consonant intervals which lie within the compass of an Octave. With the exception of the minor Sixth, which is really the most imperfect of the above consonances, the ratios of their vibrational numbers are all expressed by means of the whole numbers, 1, 2, 3, 4, 5, 6." (Helmholtz, 1895:14)

[34]See the methods of Helmholtz's investigations into sympathetic resonance (Helmholtz, 1895:36-49).

[35]Numbered or named points used to define single pitch, or frequency, ranges. Pitches may be numbered in terms of their relative vibration rate, in cycles per second, or Hertz (hz).

[36]Contiguous, diachronic, sequences and simultaneous, synchronic, relations of pitched sounds.

[37]Ascending / descending successions of notes/intervals.

[38]Limited ranges of multiple notes, simply put these are generally theorised by the use of a vertical model of pitch space. Sounds may be generally described as being in low, moderate, or high registers for a voice or instrument. This generally applies to instruments with limited ranges.

[39]The similarity of notes in various registers, despite their obvious differences of pitch height, are due to the similarity of things that are related by simple duple proportion, as is exhibited by halving or doubling a string length.

Stellenbosch University https://scholar.sun.ac.za

Attempts to analyse musical pitch space have worked with various limitations. Social expectations and functions, the listeners differing capacities for musical appreciation, the practical musical pedagogy of the musicians, these and other aspects of musical context define the limits of the musical experience and its associated avenues of expression and possibilities for experimentation. Musical canons and their pedagogies have been created in wide varieties, informed by their myriad contexts.

1.1.3.1 Semitones

Historic mathematical theories of consonances, and the manifold intervals generated by various applications of geometrically and arithmetically derived means, may be highlighted by the following comments bearing upon the definition of the semitone, an issue of no small import for practical musicians.

The standard measure of the fixed-fret semitone, as used by string instruments like the guitar, has varied widely. The frets have been fixed at various string lengths, usually between one-eighteenth and one-seventeenth of the string-length.

> The frets are wires crossing the finger-board at regular intervals, which, by shortening the string one-seventeenth of it's length, raises the pitch of the sound a semitone. (Sor *&* Harrison, 1924:7)

Hawkins (1776:108) speaking of Mersenne's instructions in his 1648 *Harmonie Universelle* describes how various forms of semitone were formerly acknowledged[40].

Hawkins (1776:181) further relates, in describing Descartes' 1617 *Musicae Compendium*, how arithmetic and geometric divisions of the octave lead to different approaches to semitone divisions. Arithmetic divisions of the octave result in varieties of semitones, while geometric division of the octave results in equal semitones[41].

[40] "In the fourth and fifth books he treats of the consonances and dissonances, shewing how they are generated, and ascertaining with the utmost degree of exactness the ratios of each ; for an instance whereof we need look no farther than his fifth book, where he demonstrates that there are no fewer than five different kinds of semitone, giving the ratios of them severally." (Hawkins, 1776:108)

[41] "Of the two methods by which the diapason or octave is divided, the arithmetical and geometrical, the author [Descartes], for the reasons contained in the sixth of his *Praenotanda*, prefers the former; and for the purpose of adjusting the consonances, proposes the division of a chord, first into two equal parts, and afterwards into smaller proportions, according to this table (see table 1.1).

The advantages resulting from the geometrical division appears in the Systema Participato, mentioned by Bontempi, which consisted in the division of the diapason or octave into twelve equal semitones by eleven mean proportionals; but Des Cartes rejects this division [...]" (Hawkins, 1776:181)

Stellenbosch University https://scholar.sun.ac.za

$1/2$	Eighth								
$1/3$	Twelfth	$2/3$	Fifth						
$1/4$	Eighteenth	$2/4$	Eighth	$3/4$	Fourth				
$1/5$	Seventeenth	$2/5$	Tenth Major	$3/5$	Sixth Major	$4/5$	Ditone		
$1/6$	Nineteenth	$2/6$	Twelfth	$3/6$	Eighth	$4/6$	Fifth	$5/6$	Third Major

Table 1.1: Descartes' Table

unit	**cents**	**interval name**
$\sqrt[1200]{2}$	1	Cent
$\sqrt[1000]{2}$	1.2	Millioctave
$\sqrt[1000]{10}$	3.99	Savart
$\sqrt[96]{2}$	12.5	Sixteenth Tone
$\sqrt[72]{2}$	16.67	Smallest step in 72-EDO
$\sqrt[53]{2}$	22.64	Holdrian Comma
$\sqrt[48]{2}$	25	Eighth Tone
$\sqrt[41]{2}$	29.27	Smallest step in 41-EDO
$\sqrt[36]{2}$	33.33	Sixth Tone
$\sqrt[31]{2}$	38.71	Smallest step in 31-EDO
$\sqrt[30]{2}$	40	Fifth Tone
$\sqrt[24]{2}$	50	24-TET Quarter Tone
$\sqrt[18]{2}$	66.67	Third Tone
$\sqrt[12]{2}$	100	12-TET Semitone

Table 1.2: Interval step sizes derived by various equable square-roots.

Hawkins (1776:181) makes further mention of the English translator of Descartes' *Musicae Compendium*[42], William Lord Brouncker, president of the Royal Society. Lord Brouncker disagreed with Descartes preference of musical divisions by arithmetical means, asserting that the geometrical was to preferred. He further proposed a division of the octave by fifteen mean proportions into seventeen semitones, and illustrated his method by algebraic and logarithmic processes.

The exhaustive research of Daniélou (1958) offers a a detailed view of the pitch relations existing within an octave. Some of the intervals between unity and a semitone are presented in tables 1.2, 1.3, and 1.4.

Such are the varieties of minutely differing, but explicitly defined and permissible, semitones, and many smaller intervals besides, that may be determined and utilised for musical effects. This brief survey of the semitone, the

[42]Published in English in 1653

Stellenbosch University https://scholar.sun.ac.za

unit	cents	prime factors	interval name
$\sqrt[11]{3/2}$	63.81	$\sqrt[11]{3} : \sqrt[11]{2}$	β scale step
$\sqrt[9]{3/2}$	78	$\sqrt[9]{3} : \sqrt[9]{2}$	α scale step

Table 1.3: Intervals derived by various equable square-roots from a base interval other than the octave

smallest basic unit of conventional pitch theory, and it's quantitative complications may serve to introduce some of the complexities of measuring intervallic possibilities. Further complications of these relations and their mathematical statement, are dealt with in chapter 3.

The laws governing pitch relations may be understood as complex periodic phenomena involving relations of interval sets and sequences, these capable of forming intersecting pitch relations, and having their basic statements induced from mathematical theory. Natural whole number relations of integers and polynomials, together with their rational number complications and the inclusion of irrational approximations may be brought into some congruence by considering dimensional extensions of theoretical pitch structures.

1.2 Aims and Objectives

The differences between the esthetic considerations of our perceptions of musical sounds and the explicit physical nature of those musical sounds are relations which this research aims to differentiate. It is considered that a relativistic physical conception, capable of mathematical statement, coexists with esthetic considerations (Helmholtz, 1895:3-10). This physical model may be explicitly defined, quantitatively, in marked contrast to the indeterminacy and contradiction arising from the many artificial frameworks of musical systems that may differ substantially when defining musical esthetics. This physical model proceeds from foundations in scientifically rooted approaches to defining musical pitch relations, and it will be argued that this model offers practical analogies for considering unseen sound relations in terms of physical quantities and spatial relations.

It is conceived that a software model of mathematical stated pitch relations may be extrapolated from various methods of existing scientific analysis in order to bear practical relations, in that these mathematical statements may be applied to a physically modeled string or some other resonant body. Such physically locatable observations are dependent upon a well defined dynamically determined model.

It is conjectured that a system of pitch relations may be explicitly defined mathematically, allowing for the retrieval and musically informative interpretation of pitch data such as would suit not only analytic needs but also

Stellenbosch University https://scholar.sun.ac.za

unit	cents	prime factors	interval name
$^{4375}/_{4374}$	0.4	$5^4 \times 7 : 2 \times 3^7$	Ragisma
$^{2401}/_{2400}$	0.72	$7^4 : 2^5 \times 3 \times 5^2$	Breedsma
$^{32805}/_{32768}$	1.95	$3^8 \times 5 : 2^{15}$	Schisma
$^{225}/_{224}$	7.71	$3^2 \times 5^2 : 2^5 \times 7$	Septimal Kleisma
$^{15625}/_{15552}$	8.11	$5^6 : 2^6 \times 3^5$	Kleisma
$^{2109375}/_{2097152}$	10.06	$3^3 \times 5^7 : 2^{21}$	Semicomma
$^{145}/_{144}$	11.98	$5 \times 29 : 2^4 \times 3^2$	Difference between 29:16 and 9:5
$^{1728}/_{1715}$	13.07	$2^6 \times 3^3 : 5 \times 7^3$	Orwell Comma
$^{126}/_{125}$	13.79	$2 \times 3^2 \times 7 : 5^3$	Small Septimal Semicomma
$^{121}/_{120}$	14.37	$11^2 : 2^3 \times 3 \times 5$	Undecimal Seconds Comma
$^{96}/_{95}$	18.13	$2^5 \times 3 : 5 \times 19$	Difference between 19:16 and 6:5
$^{2048}/_{2025}$	19.55	$2^{11} : 3^4 \times 5^2$	Diaschisma
$^{81}/_{80}$	21.51	$3^4 : 2^4 \times 5$	Syntonic Comma
$^{531441}/_{524288}$	23.46	$3^{12} : 2^{19}$	Pythagorean Comma
$^{65}/_{64}$	26.84	$5 \times 13 : 2^6$	65$^{\text{th}}$ Harmonic
$^{64}/_{63}$	27.26	$2^6 : 3^2 \times 7$	Septimal Comma
$^{56}/_{55}$	31.19	$2^3 \times 7 : 5 \times 11$	Ptolemy Enharmonic
$^{51}/_{50}$	34.28	$3 \times 17 : 2 \times 5^2$	Difference between 17:16 and 25:24
$^{50}/_{49}$	34.98	$2 \times 5^2 : 7^2$	Septimal Sixth Tone
$^{49}/_{48}$	35.7	$7^2 : 2^4 \times 3$	Septimal Diesis
$^{46}/_{45}$	38.05	$2 \times 23 : 3^2 \times 5$	Difference between 23:16 and 45:32
$^{128}/_{125}$	41.06	$2^7 : 5^3$	Enharmonic Diesis / 5-limit Limma
$^{36}/_{35}$	48.77	$2^2 \times 3^2 : 5 \times 7$	Septimal Quarter Tone
$^{33}/_{32}$	53.27	$3 \times 11 : 2^5$	Undecimal Comma
$^{31}/_{30}$	56.77	$31 : 2 \times 3 \times 5$	Difference between 31:16 and 15:8
$^{28}/_{27}$	62.96	$2^2 \times 7 : 3^3$	Septimal Minor Second
$^{27}/_{26}$	65.34	$3^3 : 2 \times 13$	Chromatic Diesis
$^{25}/_{24}$	70.67	$5^2 : 2^3 \times 3$	Just Chromatic Semitone
$^{67}/_{64}$	79.31	$67 : 2^6$	67$^{\text{th}}$ Harmonic
$^{21}/_{20}$	84.47	$3 \times 7 : 2^2 \times 5$	Septimal Chromatic Semitone
$^{256}/_{243}$	90.22	$2^8 : 3^5$	Pythagorean Limma
$^{135}/_{128}$	92.18	$3^3 \times 5 : 2^7$	Greater Chromatic Semitone
$^{18}/_{17}$	98.95	$2 \times 3^2 : 17$	Just Minor Semitone

Table 1.4: Rational Schismas, Commas, Dieses, and Semitones

Stellenbosch University https://scholar.sun.ac.za

prove of practical interest to the performing musician. By providing a specific model, of inter-related scalings, means and metrics, upon which any general vaguenesses of theoretical knowledge regarding tuning and intonation may be defined, esthetic dilemmas may be regarded qualitatively, interpreted contextually according to each instance from manifold dynamic perspectives of a unified quantitative basis.

This research goes on to propose that these constellations of tuning and temperament relations are important complimentary factors, especially useful to the description of freely pitched instruments such as unfretted strings and voices. The relation of temperaments, by logarithmic invariance, to intonations, by arithmetic invariance, may be treated as complementary factors of mathematical number theory. These relations are further examined in section 3.1.

To allow for the extraction of information regarding musical relations and the accurate determination of tonal interval structures this research investigates some of the many possibilities of tonality and interval structure. In order to define an unvarying analytic yardstick for pitch this research seeks to model a mathematical scheme of sufficient complexity to accurately mirror tonal relations, combining mathematical mean scalings representing points along the pitch continuum.

The task of cataloguing all these varieties would involve labour without end, being theoretically infinite. This research deals with a limited number of typical varieties[43].

This research aims to approach the interpretation of musical pitch in such a way that the results of this approach may be reflectively informative to the real practices of musical pitch production while also being of use to scientific researchers. This research proposes that the measure of necessary complexity required for this task should make allowances for critical inter-relation between the data of science and musical practice. This research proposes a mathematical mapping of pitch usages, in ratios and cents, against a tonally-centred generalised musical octave capable of being extended to map multi-octave melodic-harmonic pitch usages against various pitch-time lines. Against these mappings, the results of audio analysis may be plotted by correlation against sample audio data. Results may be summed as simple graphs, logged as text, or incorporated into spreadsheet data graphs. Specific mathematical identities may be related to terminological uses by correlation with further dictionary definitions. In particular, this research seeks to investigate the ways in which various intonations and temperaments may be practically gauged and realised in a complimentary framework of mathematically explicit musical theory and common terminological usage.

[43]Namely, rational intonations up until the 7-limit, and irrational temperaments up until 53-EDO.

Stellenbosch University https://scholar.sun.ac.za

Chapter 2

Literature Review and Problemata

[T]he idea of tuning invariance, by which relationships among the intervals of a given scale remain the "same" over a range of tunings [...] requires that the frequency differences between intervals that are considered the "same" are "glossed over" to expose underlying similarities.

– (Milne et al., 2007:15)

A brief history of scientific and musical approaches to the technical analysis and modeling of the relations of musical pitch will be presented in the course of this study. These will be discussed in terms of the problems that they seek to elucidate.

2.1 Invariance and Tonal Dynamics

2.1.1 Invariant Relations of Frequency

In considering the mathematical dimensions of similarity by which pitch relations may be shaped, it serves us to make use of some invariant scale/s of measure by which we may treat pitch relations as we do families of mathematical variety. These similarities may be determined from observed relations of mathematical mean proportions.

The various similarities that define pitch invariances are described by the proportions of the geometric, arithmetic and harmonic means as they are generally theorised by mathematics[1].

[1] Arithmetic means are points arranged in such a way that they construct equally sized, and unequal proportioned, parts. Geometric means are points arranged in equal proportions

Stellenbosch University https://scholar.sun.ac.za

Each of these means describes some relation of similarity. These relations may be further 'mingled', unified by their translations to each other, in order to be made to furnish perspectives of their relative differences along a theoretical continuum, such as the range of pitch.

These invariantly 'attuned', or proportioned, pitch structures allow for the relatively simple statement of complex pitch shifting phenomena, and also allow for explicit analysis of pitch varieties using a variety of tuning and temperament schemes defined in terms of mathematical relations.

> [T]uning invariance can be a musically useful property by enabling (among other things) dynamic tuning, that is, real-time changes to the tuning of all sounded notes as a tuning variable changes along a smooth continuum. (Milne *et al.*, 2007:15)

Ideas of tuning invariance, coming as they do from a computational perspective of intonational investigation, are most often modelled as *button-based*[2]. methods of musical intonation. In these a limited number of discretely pitched sounds are accessed, and reproduced, by a keyboard layout with some assumed tuning scheme, and provision for methods for fixed-pitch adjustment. This last novelty allows for the scalar redistribution of pitches, by which otherwise complex redistributions of pitch may be modeled simply, and indicated computationally, by singular adjustments to the values of single variables.

In the case of modulations within temperament systems, such as modulations within 12-TET, the seven pitches of the diatonic major scale and their chromatic relations are isomorphically invariant throughout the system for all available root pitches. The pitches of this diatonic major scale may be adjusted throughout a number of temperaments and tuning schemes by changing the mean proportion governing their distributions. Within this general framework any adjustment to the underlying mean proportions describing pitch distributions result in shifts to pitch proportions, their number and location, and greatly effect any associated pitch set theory.

2.1.2 Tuning and Temperament Variations

> The object of temperament [...], is to render possible the expression of an indefinite number of intervals by means of a limited number

of unequally sized parts. Harmonic means are points arranged by progressively unequal proportions and similarly unequal parts (Barker, 1989:42).

[2] "A button is any device capable of triggering a specific pitch; it could be a physical object such as a key or lever, or it might be a 'virtual' object such as a position on a touch-sensitive display screen or in a holographic projection. A layout is the embodiment of a temperament in the button-lattice of a musical instrument." (Milne *et al.*, 2007:7) Button-based systems, such as traditional piano keyboards, accordion buttons and other isomorphic keyboard designs, are described by Keislar (1988:3-6), and Milne *et al.* (2008).

Stellenbosch University https://scholar.sun.ac.za

> of tones without distressing the ear too much by the imperfections of the consonances. The general practice has been from the earliest invention of the keyboard of the organ to the present day to make 12 notes in the Octave suffice. This number has been in a very few instances increased to 14, 16, 19, and even to 31 and 53, but such instruments have never come into general use. (Helmholtz, 1895:431)

The broad differences of tuning systems, which result from their use of the different mathematical means, are those of their scaling. Arithmetic means alone are preferred by advocates of *just* Pythagorean and Ptolemaic tunings[3], while geometric means alone are preferred by advocates of EDO tunings[4]. This research will suggest collaborative uses of these means and their scaling.

Reconciliation of these respective methods has been a source of great division amongst theorists. Although the practice of representing temperament schemes by deviation from 12-TET intervals has become standard practice it requires the reduction of pitch relations to a decimal logarithmic notation. This last being somewhat unwieldy and counter-intuitive to the consideration, elaboration and practical representation of periodic relations.

The use of both arithmetic and geometric means may be justified by a consideration of the underlying number theory.

Dedekind, in his remarkable essay on continuity and irrational numbers, draws the definition of a line as a continuity which may be divided into such a way that a single point, a cut (*shnitt*), divides the line into two *classes* of points, respectively those points above the cut and those below. In such a way this method allows for the division of a line into classes by both rational and irrational number points. He does however stress the critical difference between rational and irrational numbers. Rational numbers refer to integral relations and model material constructs through finite quantities. Irrational numbers conversely model infinite quantities, which may never be explicitly realised in finite representation except by approximation to those finite points nearest the infinite cut representing the irrational number. (Dedekind, 1901:6-19)

Irrational number sets (of cuts) correspond to those relations which are sought for in describing systems of *equal-temperament*, so named because of their geometric scaling and resulting equivalence of interval proportions in terms of pitch-space. Rational number sets of cuts correspond to those relations which are sought for in describing systems of *just* tuning varieties, including Pythagorean, and Ptolemaic tunings, and deal with arithmetic divisions of pitch space and sounding bodies.

Both schools of though have their adherents. Helmholtz accorded just tunings a more prestigious status justifying this on the strength of the combi-

[3] See Duffin (2008), Bosanquet (1874), and Helmholtz (1895).

[4] See Mandelbaum (1961), Cho (2003), and Blackwood (1991).

Stellenbosch University https://scholar.sun.ac.za

national tones produced by such systems (Helmholtz, 1895:197-211), though accepting that there were some unique challenges involved in modulations of such just tonalities (Helmholtz, 1895:234-235).

The difficulties of these challenges are especially important issues of concern for instruments having fixed pitches, as a measure of compromise often needs to be introduced in respect of the number of desired and available fixed pitches per octave.

Such tuning schemes as suit the needs of fixed pitch instruments may be irregular, well-tempered schemes, or regular, mean-comma, and equally tempered, but are almost always inevitably *cyclic*, containing exact octave replications for each pitch in every octave. In considering the uses and formations of non-cyclic temperaments and intonation, the model of a string fixed from bridge to bridge, represented visually as a line and its linear divisions, by its simple relations to mathematical principals, is inherently well suited to such descriptions of pitch space.

2.1.2.1 Equality

> The system which tuners at the present day intend to follow, though none of them absolutely succeed in so doing [...], is to produce 12 notes reckoned from any tone exclusive to its Octave inclusive, such that the Octave should be just and the interval between any two consecutive notes, that is, the ratio of their pitch numbers, should be always the same. This is known as Equal Temperament [...]. The interval between any two notes is an Equal Semitone, and its ratio is $1 : \sqrt[12]{[2]} = 1 : 1.0594631$, or very nearly $84 : 89$. If we further supposed that 99 other notes were introduced so as to make 100 equal intervals between each pair of equal notes, these intervals would be those here termed Cents, having the common ratio $1 : \sqrt[1200]{[2]} = 1 : 1.0005778$, or very nearly $1730 : 1731$. (Helmholtz, 1895:431)

A cent[5] is the standard unit of equally-tempered (ET) measurement, serving especially well for 12-TET, there being 100 cents to each 12-TET semitone and 1200 cents to the ET octave. The proportions of scaling indicated by the unit of the *equal* cent, serving in translation to and from frequency number, has since its first description become a veritable standard for unit scale in the scientific literature.

Such equal divisions of scale have an important place in the history of mathematics.

The 12-TET tritone is represented mathematically by the irrational number, $\sqrt{2}$. The solutions to such irrational formulas have been approximated in

[5] Described above by Ellis, and further elaborated in Helmholtz (1895:446-451).

Stellenbosch University https://scholar.sun.ac.za

the classical literature of mathematics of many cultures with various measures of precision and have ever been held to be remarkable for the way they model the mathematics that they represent.

> Baudháyana's Súlbasútra gives a remarkable approximation to $\sqrt{2}$:
>
> $$\sqrt{2} = 1 + \frac{1}{3} + \frac{1}{3\times 4} - \frac{1}{3\times 4\times 34} = \frac{577}{408}$$
>
> This is accurate to five decimal places. It is intriguing that Baudháyana felt the need to add the last term in the expansion because without that the approximation is still valid to three decimal places and excellent for geometric constructions. (Kak, 2009:3-4)

The necessary approximation of these irrational numbers, as fractional compounds, results from the infinitely divisive implied quantities of irrational number sets. Infinite quantities implicitly resist exact statement by finite number relations.

It has been asserted by many musicians and musicologists that the logarithmic deduction of equal-tempered pitch relations, and the twelve tone (12-TET) model of musical structure, entail compromises of desired harmony[6] (Helmholtz, 1895:310-371). These compromises offer a simple convenience for certain instruments[7]; however, by the systematic avoidance of real number pitch relations, these systems obscure the rational mathematical relations of musical sounds as related by the historically complex harmonic rationale of rational number sets asserted by many authors and musicians such as the elder Mozart (1985:108), Sir John Hawkins (1853:600-660), Baron Helmholtz (1895:234-309), and many classical Greek authors from Aristoxenus to Ptolemy.[8]

The scientific determination of logarithmic cents and their analytic relations to perceivable tonal phenomena is an issue of no small importance. Notwithstanding the necessarily approximate points that these logarithmic means represent, they may feature usefully in the translation and correlation of different systems of temperament. Together with various schemes of arithmetically mean combinations of tones, these may be correlated to form a usefully full diagram of the octave space[9]. The construction of an invariant pitch framework is a useful yardstick by which to measure tonal relations; however,

[6] See Duffin (2008) and Busoni (1911) amongst others.

[7] Such as those having sounds activated by *button pressing*, the button's location or preparation serving to determine the pitch of the sound, including in this definition all keyboard layouts and fretted instruments. See Ellis (1863*b*,*a*).

[8] See Barker (1989) and West (1992).

[9] Such a scheme is further related in Chapter 3.

Stellenbosch University https://scholar.sun.ac.za

it does not by itself indicate the possibilities inherent in tonal relations[10]. The scale of pitch is not in itself a *self organising map*(SOM) of musical sounds, it is merely a surface upon which we may impose the great many varieties of tonal landscapes. Such topological relations as equalities, logarithmic and arithmetic respectively, allow for the plotting of many dynamic varieties of temperament schemes and intonations as relational maps defined by integral and enharmonic rational numbers and approximate irrational number identities.

2.1.3 Just Intonation

2.1.3.1 Complexities of Simple Number Relations

It has been posited historically that music relations are defined by integral[11] and enharmonic[12] ratios, and that only these are themselves capable of supporting a scheme of rational harmonic relations (Helmholtz, 1895:250-290). The mathematical relations of sounds by rational numbers offer a complex, and historically informed, model of pitch relation, usefully describing instances of tonal usage, generally and in specific contexts. The complexity of this scheme is capable of describing tonal dynamism in terms of accurate pitch ratios and deviations (Helmholtz, 1895:272, 290-306).

The perfection of pure integer and enharmonic relations, and of exacting equality for that matter, is for the most part eluded by the coarseness of our performing conceptions and abilities. However, such relations may still be attained (Helmholtz, 1895:325), and may form not only the basis of our performing intentions and conceptions but come to contribute features of key importance to musical pitch theories.

Hugo Riemann's late essay, *Ideas for a Study 'On the Imagination of Tone',* identified enharmonic equivalence as a theoretical puzzle of the greatest importance, stating that "*the study of enharmonic identification [...] ultimately will solve and explain the contradictions between the results of tone-psychological investigations and the practical experiences of musicians.*" (Wason *et al.*, 1992:110)

Rational number set models of musical intervals are highly complex to the mathematically uninitiated musicological interpreter. Numbered limits and

[10]This follows on the reasoning presented by Aristoxenus in his argument against *Katapyknosis*(Barker, 1989:106, 154). A whole may be theoretically divided into as many parts as is necessary. It does not necessarily follow that all of these divisions are required to be used at once, or in direct sequence. So too one may seek for as many divisions as are necessary to indicate an invariant logarithmic or arithmetic scale, such often being desirable for its practical utility, however, it does not then follow that we are to make use of all the resulting divisions, at once, or in direct sequence.

[11]Simple number relations, or integer relations.

[12]Compound or polynomial number relations.

Stellenbosch University https://scholar.sun.ac.za

their fractional ratios reveal complications of great depth, especially amongst the enharmonic relations of polynomial number relations.

The simplest musical instance of the rational number identities involves the doubling and halving numerical values of the octave. 3-limit intervals, further discussed below in section 2.1.3.2, arise from the interplay of the numbers two and three and those of their multiples. Further elaborations of these rational interval schemes incorporate progressive prime numbered limits. The considerations of extended sets of such mathematical limits of intonation quickly lead to a profusion of intervals, even when excluding octave equivalences. Helmholtz noted, in these relations, attendant acoustic phenomena constituting combination tones and resonant reinforcement.

Complexities of close enharmonic relations and Aristoxenian theories of melodic succession and non-melodic adjustments provide interesting comparisons in this connection. The laws given by Aristoxenus for melodic succession[13] and his descriptions of the 'compressions' of pitch constituted by very small intervals are capable of modelling by software and may come to serve as a basis for 'melodic' and 'non-melodic' dictionaries of intervals (Barker, 1989:170-182).

2.1.3.2 n-limits

> [Vedic] texts are associated with a numerical mysticism in which the many (numbers) emerge from the one (number) which is why properties of numbers were of interest [...] the Vedic authors connected the number of divisors to certain periodic processes. (Kak, 2009:10-11)

In the projection of arithmetic means, such as govern just tuning schemes, variously numbered limits are used in music to define the derivation of pitch locations by the projection of a periodic function.

The generation of musical intervals, as sets derived from the interplay of number limits, proceed in the same order as the prime number series.

The '3-limit' is the limit adhered to by the allegedly 'Pythagorean' system of intonation, and consists of the set of number relations that may be derived from multiples of the prime odd number three and the even number two. Though these products are commonly limited at 729 (3^6) and 1024 (2^9) they are seen to form a divergent series which may continue infinitely, closing only approximately in the form of very small intervals, variously known as schismas, and commas. Resisting the simple cyclic arrangement of temperaments upon a circular octave, this scheme may be commonly arranged in the form of a spiralling cycle of fifths of ratio $^3/_2$.

[13] See section 3.2.2.

Stellenbosch University https://scholar.sun.ac.za

The 5-limit[14] contains the set of number relations that may be formed from all multiples of the numbers, two, three and five. This system is noticeably more complex than the preceding 3-limit, as it contains all of the three limit intervals as well as a great many syntonic-comma ($^{81}/_{80}$) distant enharmonic equivalences. This system also introduces other enharmonic equivalences such the greater and lesser diesis, arising from the cycling of major thirds of $^{5}/_{4}$, as augmented traids, and minor thirds of $^{6}/_{5}$ as diminished quartads. Each augmented cycle of three major thirds falls short of the octave by a lesser diesis of $^{128}/_{125}$, and each diminished cycle of four minor thirds exceeds the octave by a greater diesis of $^{648}/_{625}$. This last is itself a composite interval consisting of a lesser diesis joined to a syntonic comma.

The 7-limit is the limit of the Septimal intervals, those relations arising from all combinations of the multiples of the numbers, two, three, five and seven. With the introduction of the number seven this system again multiplies all the complication of the limits that came before it, offering a vast array of commas and enharmonic equivalences, as well as some very unique intervals such as the 'tone' of $^{8}/_{7}$ and the 'minor third' of $^{7}/_{6}$. This system furnishes some very useful correspondences to the overtone series such as the interval of $^{7}/_{4}$ as a form of 'seventh', and the tritones of $^{10}/_{7}$ and $^{7}/_{5}$.

Numbered limits, drawn from the series of prime numbers, may be pursued as far as needs demand, with each successive numbered limit adding to the compounding of the previous intervals generated.

2.2 Historic Approaches to Musical Mathematics

> [...] in this [...] part of our inquiry into the theory of music we have to furnish a satisfactory foundation for the elementary rules of musical composition, and here we tread on new ground, which is no longer subject to physical laws alone, although the knowledge which we have gained of the nature of hearing, will still find numerous applications. We pass on to a problem which by its very nature belongs to the domain of esthetics. (Helmholtz, 1895:234)

2.2.1 Linking Sensation to Perception

Considerations of esthetic laws are necessarily subject to great divergence. Historic developments and social taboos bear equal weight upon the problems of esthetics and artificial systems. Musical theories, and the phenomena of sound

[14] In this research this limit is termed the *Ptolemaic* system.

Stellenbosch University https://scholar.sun.ac.za

that underlies the sensations of musical tone and pitch, may be investigated for their relations[15].

An important fact may be established upon these observations. It is not the phenomena of sound, but the social perceptions of those sounds and their 'affect', or representations, that result in such variable interpretations[16].

Speaking in terms of musical pedagogy, without mathematical description, risks eluding the specific which we attempt to describe. Associating such necessarily general terminology with specific mathematical description helps to retain exact description, and furnishes a compound structure capable of encompassing potential pitch structures and their possible contexts.

2.2.2 Greek Mathematical History

A mathematically explicit methodology of musical pitch practice was historically pioneered by ancient Greek mathematicians.

The existing fragments of ancient Greek writings that deal with theoretical models of music consist chiefly of mathematical axioms and their relation to sonorous string divisions[17]. The linear analogy of string divisions is further understood to be representative, and in imitation, of the movements of a voice. The recorded fragments of these writings make many useful remarks in

[15]"[P]reviously, in the theory of consonance, of agreeable and disagreeable, we referred solely to the immediate impression made on the senses when an isolated combination of sounds strikes the ear, and paid no attention at all to artistic contrasts and means of expression; we thought only of sensuous pleasure, not of esthetic beauty. The two must be kept strictly apart, although the first is an important means for attaining the second.

The altered nature of the matters now to be treated betrays itself by a purely external characteristic. At every step we encounter historical and national differences of taste. Whether one combination is rougher or smoother than another, depends solely on the anatomical structure of the ear, and has nothing to do with psychological motives." (Helmholtz, 1895:234)

[16]"But what degree of roughness a hearer is inclined to endure as a means of musical expression depends on taste and habit; hence the boundary between consonances and dissonances has been frequently changed.

Similarly Scales, Modes, and their Modulations have undergone multifarious alterations, not merely among uncultivated or savage people, but even in those periods of the world's history and among those nations where the noblest flowers of human culture have expanded.

Hence it follows, - and the proposition is not even now sufficiently present to the minds of our musical theoreticians and historians that the system of Scales, Modes, and Harmonic Tissues does not rest solely upon inalterable natural laws, but is also, at least partly, the result of esthetical principles, which have already changed, and will still further change, with the progressive development of humanity." (Helmholtz, 1895:234-235)

[17]As seen in Barker (1989) and Creese (2010), these models are practically exhibited upon the monochord.

Stellenbosch University https://scholar.sun.ac.za

terms of both physical observations of quantity and more qualitative esthetic description.

As Macran informs us in his introduction to Aristoxenus' Harmonics:

> "In the music of Ancient Greece we are able to trace, though unfortunately with some gaps, the first steps of [...] a [*mathematical*] development. The earliest students of the science, in endeavouring to establish a scale or system of related notes, started as was natural from the smallest interval, the bounding notes of which afforded an elementary relation. This they found in the interval of the Fourth, in which the higher note is tonic ; and this melodic interval, essentially identical with our concord of the Fifth, may be regarded as the fundamental sound-relation of Greek music. When they had thus secured a definite interval on the indefinite line of pitch, their next concern was to ascertain at what points the voice might legitimately break its journey between the boundaries of this interval." (Aristoxenus *&* Macran, 1902:5-6)

The principles of Euclidean geometry, as used for defining abstract general statements of mathematics, bear important relation to the physical manifestations of sound and its associated sensations. Further treatment of Euclidean number families, such as epimoric[18] and epimeric[19] relations, are used in classical Greek musical theory to assist in categorising the number relations of pitch (Barker, 1989:196-7).

The descriptions of the various means in music, as given by Archytas[20], are of great importance to these pitch theories. According to this description there are three means in music, the arithmetic, the geometric and the harmonic[21]. Arithmetic means are points arranged in such a way that they construct equally sized, and unequal proportioned, parts[22]. Geometric means are points arranged in equal proportions of unequally sized parts[23] Harmonic means are points arranged by progressively unequal proportions and similarly unequal parts[24].

Plato describes a musical model represented as a great spinning whorl, around which is wound a stretched string. This string is divided in half and

[18] $n + 1 : n$

[19] $n + m : n$, where $m > 1$

[20] In Barker (1989:42).

[21] Or 'subcontrary'.

[22] Such means are used in music for the rational divisions of strings and other sonorous objects.

[23] Such means are used to measure invariant scales, such as that of Frequency Numbers in Hz.

[24] Such means, as are determined by progressive epimoric and epimeric relations of integer ratios, are used to describe harmonic relations.

Stellenbosch University https://scholar.sun.ac.za

the resulting ends of each part are then connected, forming two circles, each an identical representation of an octave. One of these is then set inside the other. These circles he termed the *same* and the *different*, respectively[25]. The continuation of his narrative incorporates the apocryphal figures of the fates, daughters of necessity[26]. These may be interpreted as representing successive diachrony, simultaneous synchrony, and precedent diachrony. These admittedly poetic images are mentioned for the curious similarity they offer to the notions of a dynamic tonal scale and related phenomena (Barker, 1989:58-61).

2.2.2.1 Tetrachord Systemas

Methods for the construction of musical scales from tetrachord systems may be elucidated from various scattered fragments and notable writings of classical Greek authors[27].

It follows from their rationale that, as the hand is capable of grasping four notes at once, or in one position, upon a string instrument such as the classical Greek lyra[28], so this form deserves special regard, as it is at once accessible and capable of 'animation'(Barker, 1989:375). This form is the tetrachord, and the varieties of its arrangement ground the Greek theory of musical pitch. The interval of the fourth, $^4/_3$, was accorded special significance in their theories as being the smallest of the consonances, within which one may find exemplary samples of the variety which constitute larger tonal structures called genera,

[25]These term represent similarities to those relations of pitch described in classical Greek theory by the terms *thesis* (lit. 'placing') and *dynamis*(This is related to the philosophic idea of *dunamis* 'potentiality'. For further discussion of this term, see De Groot (2008).). In relation to pitch placement, these terms refer to the practical position and the dynamic potential of a pitch, respectively. Practical positions are such relations as constitute direct representations of practical pitch placements, as divisions of a string or some other sounding length, while a dynamic potential indicates only the relations of pitch. Though theoretically utilising the same scale and framework, practical relations of *dynamis* / dynamic potential may make use of the relations of *thetic* / practically located relations, and, depending on the modulatory tonal framework and mechanisms of tone production, these relative frameworks undergo some superposition. For example, upon a string taken to be unity, or $^1/_1$, the fifth diatonic pitch of dynamic potential $^3/_2$ will indeed be found at the thetic value of $^2/_3$rds of the string length, and the 'twelfth' of dynamic potential $^3/_1$ will be found at the thetic value of $^1/_3$rd of the string length. However, the fifth diatonic pitch of dynamic potential $^3/_2$ may also be found at the thetic value of unity, or $^1/_1$, upon a string tuned a fifth higher than the initial string, where the 'twelfth' of dynamic potential $^3/_1$ will be found at the thetic value of half($^1/_2$) of the string length. These superpositions become increasingly complex with the addition of further string-length relations and modulation tonal systems. This small example has given a brief explanation of how thetic position and dynamic potential were seen to differ in practice by classical Greek theory, for more on this see Barker (1989).

[26]Lachesis, who sings of what has been; Clotho, who sings of what is; and Atropos, who sings of what will be.

[27]These may be found, in translation and with voluminous footnotes, in Barker (1989).

[28]An ancient Greek form of bowed instrument with three strings tuned in 'fifths'.

Stellenbosch University https://scholar.sun.ac.za

many of which are similar to those of modern scales, modes and tonal keys (Barker, 1989:350-1).

These fall into three broad varieties, the diatonic, the chromatic, and the enharmonic. The usage of these terms is related to the modern usage, however holding notable distinctions. Diatonic tetrachords are constituted from the relations of two large steps and a small step, with one exception of three large steps, chromatic tetrachords are constituted from two small steps and a single larger step, larger than the diatonic large step, while enharmonic tetrachords contain two very small steps, and a single much larger step. These last are respectively the smallest and the largest step used within the range of the tetrachord. The small step above corresponds to a generalised semitone, the large step to a generalised whole tone, the larger step to a generalised augmented tone, the largest tone to a generalised ditone, and the smallest step to a generalised 'quarter'-tone, or microtone. Amongst these it is remarked that the diatonic has three varieties, the chromatic has two varieties, and the enharmonic has but a single variety of tetrachord. To these I would add that there are a few more intermediary and composite forms such as are greater or less than the span of a fourth yet constitute a series of four sounds divided by three intervals.(Barker, 1989:350, 140-147)

The first diatonic tetrachord variety is those of two large steps (generalised 'whole tones') followed, or preceded, in succession by a smaller step (a generalised 'semitone'), and is respectively major sounding (modern Ionian mode) or very minor sounding(modern Phrygian mode). This form is the more remarkable for being, to the Greek authors at least, similar whether ascending or inverted in descent[29]. The second diatonic tetrachord variety is that featuring a smaller step with one larger step either side, notable for its symmetry. The third diatonic tetrachord variety is that which strictly speaking exceeds the fourth, of $^4/_3$, this tetrachord consists of three large steps, constituting the tritone[30].

An exhaustive treatment of tetrachord forms is given in Chalmers (1993), and much of the preliminary Euclidean mathematics is available in Barker (1989) together with a thorough treatment of many classical Greek texts dealing with classical harmonic science.

From the varieties of the tetrachords there arise the varieties of the genera of octave scales, the 'modes' of the ancient Greeks, given systematic form in their writings. The forms of the tetrachords may be built into genera and systems of fifths and octaves and greater spans. These may be variously modulated by varying their disjunction and conjunction, and may undergo further

[29] Arguably, the Greeks understood such elaborations of inversions pragmatically. As they so often neglected such issues entirely, it seems that they treated such elaboration as assumed.

[30] This is the first tetrachord of the fourth mode (Lydian mode) of the modern major scale(Ionian mode).

Stellenbosch University https://scholar.sun.ac.za

permutation by interchanging of their varieties.

The various genera of Ptolemy illustrate many possible varieties of diatonic interval progression, and are also notable for his approval of septimal interval compounds (Barker, 1989:350-351). This illumination of various kinds of simple interval combinations, when combined with the *Harmonics* of Aristoxenus (140-147 Barker, 1989:152-169), and his laws of melodic succession[31] (Barker, 1989:170-182), form a complex approach to the problem of pitch relations incorporating mathematical ideas that greatly assist the terminological description of pitch phenomena.

Nicomachus's collection of the various systema of the tetrachords offers a glimpse into the changing nature of the Ptolemaic systemas in the later classical age. His specific use of terms such as *note, interval, relation* and *difference*[32] are also useful for defining the different contextual meanings and usages of pitch relations(Barker, 1989:266-269).

2.2.3 Pitch Space

> Investigations into how localization in the visual field comes to pass have led [...] to reflect[ions] on the origins of spatial intuition in general. This leads first of all to a question whose answer definitely belongs to the sphere of exact science, namely, which propositions of geometry express truths of factual significance and which, on the contrary, are only definitions or consequences of definitions and their particular manner of expression? [O]ne could follow this direction and find out which analytical characteristics of space and spatial magnitudes must be presupposed in order to ground the propositions of analytic geometry completely from the beginning[33]. (Pesic, 2013:276-277)

The notion of pitch-space is intrinsic to any modeled topography of pitch data. This despite the fact that the spatial analogy offered up for pitch (e.g. high and low notes), is not strictly observed, but rather induced and modeled in metaphor as an analogy.

2.2.3.1 Spatial Intuition

In music we often deduce specific principles from generally assumed principles. These assumed principles have had their general statements induced from some specific principle observations and have been proven by their practical relevance over time. Without due consideration given to the principle

[31] See section 3.2.2.

[32] *Pthongos, Diastema, Schesis*, and *Diaphora* respectively.

[33] Hermann Helmholtz, "Ueber Die Thatsáchlichen Grundlagen der Geometrie," [1866]; translated as "On the Factual Foundations of Geometry," in Pesic (2007:47)

Stellenbosch University https://scholar.sun.ac.za

assumptions that have been arrived at by the complexity of historic development, the reasoning behind any specific analysis may be subject to the inherent shortcomings of such generally assumed principles.

When we speak of a high pitch or a low pitch, to what spatial analogy are we directing this metaphor? A *high* musical tone, as a periodic undulation produced by some source of sound, is not visually observed above a *low* musical tone, though such a relation is often perceived. Stemming from the same location, these sound waves travel by transverse waves, and may be more accurately be said to be *enfielding*, and *enfielded*[34] by, each other. While a scalar metaphor may be used as a linear scheme for representing information regarding pitch-height in terms of longitudinal waves, such longitudinal depictions are representative of a phenomena that is seen to act through transverse wave motion.

2.2.3.2 Metric Analogies

> In the Western musical tradition, two pitches are generally considered the "same" if they have nearly equal fundamental frequencies. Likewise, two pitches are in the "same" pitch class if the frequency of one is a power-of-two multiple of the other. Two intervals are the "same" (in one sense, at least) if they are an equal number of cents wide, even if their constituent pitches are different. Two melodies are the "same" if their sequences of intervals, in rhythm, are identical, even if they are in different keys. Many other examples of this kind of "sameness" exist. It can be useful to "gloss over" obvious differences if meaningful similarities can be found. (Milne *et al.*, 2007:15)

Upon such similarities and congruences schemes of mathematical analysis may be framed. These schemes may conceivably frame variously complicated relations by the reconciliation of their metric topologies.

> The independence of the congruence of rigid point-systems from place, location, and the system's relative rotation is the fact on which geometry is grounded. [35] (Pesic, 2013:278)

Metric relations may construct a variety of conceivable systems of relations, these understood within frameworks of spatial analogies. Framed thus, metric topographies may be seen to converge or diverge as do shifts of perspective. Embedding any topological space within a co-ordinate system relates their frames of reference, such embeddings frame analogous resemblances.

[34] To borrow the terminology from Gomez, a character created by Kornbluth (1970).

[35] Helmholtz, "Ueber die Tatśachlichen Grundlagen", pp. 616-617; "On the Factual Foundations,", pp. 50-51 in Pesic (2007).

Stellenbosch University https://scholar.sun.ac.za

2.2.3.3 Chromoscale Analogies

The differences between the sensations of sound and the sense of sight, in regards to the colour analogy, has been a rich source of musical philosophy, having been investigated by such philosopher-scientists as Sir Isaac Newton[36] and Goethe[37], and it may serve us well to investigate how such an approach was treated by successive acoustic researchers.

> An important attribute of a musical composition is its "chroma", defined first in Ancient Greek Music[38]. Apart from the separation in "chromatic" and "diatonic" entities, as defined in Western music, there are also musical phenomena, which cannot be categorized in this way, such as Oriental music, Byzantine music and prosodic vocal phenomena[39]. (Politis *&* Margounakis, 2003:1)

Chroma, in its sense as colour, and the resulting term chromatic is used in this research in the same way it was used in Greek theory. Namely, representing any tone, interval or proportion that is not represented in the diatonic distribution of notes, and intervals, into the familiar heptatonic scale and its attendant modes.

Transpositional invariance of pitch relations is realised by the perception of moveable, invariant, relations in tuning and intonational practices. This is quite different to the scheme of colour relations, which do not offer such transpositional invariances, nor does colour modulate in the same manner as does sound[40]. The colour analogy of music, realised in the term chromoscale, is to be strictly understood as an artificial relational scheme.

2.2.3.4 Mathematically Framing Tuning Systems

> As Vogel notes, Helmholtz's "exposition of the theory of the tone quality of musical instruments was essentially grounded in mathematics"; [41] (Pesic, 2013:279)

The mathematical relation of tuning systems may be usefully reduced to the relations of simple generators and their multiplicands, define as many generators as are necessary.

> Any just interval is expressible as the product of powers of prime numbers:

[36]See Newton (1965).

[37]See Von Goethe *&* Eastlake (1970).

[38]West (1992)

[39]See Politis *et al.* (2002).

[40]See Itten (1973).

[41]See Vogel (1993:273).

Stellenbosch University https://scholar.sun.ac.za

> $2^a \times 3^b \times 5^c \times ... \times L^n$,
> where a through n are integers and L, called the "limit" of the system, is the largest prime number in the series. As an example, a 7-limit just intonation might include the ratio $^{14}/_{9}$ which reduces to
> $2^1 \times 3^2 \times 5^0 \times 7^1$. (Keislar, 1988:2)

Pursuing 5-limit intervals in this manner provides a relatively simple triple-generator interaction field. These three generators provide for the projection of many different interval tunings. Adding a further generator of 7-limit interactions yields exponentially more projections of interval possibilities, and describes the limit of the integral prime generators, all further prime generators being polynomial.

This system of interval projection may also be expanded to include more *irrational* temperament schemes and their relations. These may be marked by their 'error' from the rational intervals of the various limits, as each temperament may be seen as containing a number of valid approximations to various intervals formed amongst the n-limits. Such deviations from rational intervals are termed 'errors' after Bosanquet's usage. Conversely, rational tuning schemes may also be compared to irrational temperament schemes in terms of 'deviations'. (Bosanquet, 1874:391)

> If a (non-JI) regular tuning is created by changing the value of G_2, the value of the fifth, and all other intervals, changes in a patterned way. Assigning the magnitude of one or more of the generating intervals to a control interface provides a convenient means to "navigate" the tuning continuum. Particular values within this continuum may produce some intervals that approximate JI intervals and so are rationally identifiable; we might consider, therefore, that there has been a mapping of JI intervals to that tuning system. (Milne *et al.*, 2007:20)

Such considerations of rational tuning system relations, as mappings to which irrational temperament schemes sometimes approximate, may help to show how some temperaments and certain of their intervals come very close to modelling rational tuning relations.

> For such a temperament-mapping to be transpositionally invariant, it must be linear, though it need not be invertible (i.e., it need not be one-to-one). The embodiment of such a temperament-mapping in a suitable tuning system is called a regular temperament, and it can be characterized by the small JI intervals called commas that are tempered to unison [See Smith (2006).]. This means that a regular temperament is characterized by its temperament-mapping,

Stellenbosch University https://scholar.sun.ac.za

> not its tuning, so any given temperament has a range of suitable tunings. (Milne *et al.*, 2007:20)

Tuning systems are determined by the multiplicative relations of its intervals. These are ranked by their relations to prime number limits[42], and the multiplicative dependence of their integer relations.

> To be concrete, two or more intervals a1, a2, . . . , an are said to be multiplicatively dependent if there are integers $z_1, z_2,\ldots,z_n$, not all zero ,such that $a_1{}^{z1}\ a_2{}^{z2}\ \ldots a_n{}^{zn} = 1$. If there are no such z_i, then the a_i are said to be multiplicatively independent. (Milne *et al.*, 2007:20)

The rankings of tuning schemes are determined by the number of multiplicative independent values inherent in their definition. Temperament-mappings may be ranked by their approximations to tuning systems.

> The rank of a tuning system is the number of multiplicatively independent intervals needed to generate it. A regular temperament typically has lower rank than the JI system that is temperament-mapped to it (i.e., the mapping is non-invertible). When the temperament-mapping loses rank, all intervals can no longer be just. However, as long as there is a range of generator values over which the intervals are correctly rationally identified, the regular temperament can be considered to be valid. This is analogous to the way a projection of the three-dimensional surface of a globe to a two-dimensional map inevitably distorts distance, area, and angle. However, so long as the countries have identifiable shapes, the projection can be considered valid. Different map projections result in different distortions, and some map projections are more or less suitable to specific purposes. Some projections (such as the Mercator Projection) have the virtue of wide familiarity; so it is also with temperament-mappings (such as those that lead to 12-tet). (Milne *et al.*, 2007:20)

The inter-relation of these various schemes of temperament and tuning, stated mathematically as relative mappings, provides a method for investigating the varieties of pitch differences and similarities that may be thus described. When projecting temperament mappings to tuning schemes, provision for well-defined tolerances enables useful approximations to higher ranking identities to be defined in relation to exact generator functions.

The mathematical definitions of tunings and temperament schemes and their relative similarities and differences, are dealt with further in section 3.3.

[42]See section 2.1.3.2.

Stellenbosch University https://scholar.sun.ac.za

2.2.4 Tones, Tonics and Tonalities

It is difficult to define the word *tone* by any singular meaning. This term is synonymous with individual musical sounds (musical pitches), describes certain gauges of pitch intervals, and is also used to refer to more diffuse meanings and contexts.

Tonal descriptions of intervals express relative tensions[43]. These tensions are grouped into families of tonalities by Western classical theory, the same are termed '*gramas*' in Indian classical music (ICM). These are described as scale degrees, using seven interval / note names[44]. Combined with five altered versions of these interval / note names [45] these names describe twelve general scale degrees[46]. These scale degrees usefully express 'horizontal' melodic successions of sounds and 'vertical' harmonic sounds occurring simultaneously. These degree names may describe intervals, as well as each individual pitch forming the boundaries of intervals.

2.2.4.1 Tonality and Tonic

> The term tonality was popularized by Fétis in the early nineteenth century, and he provided a useful, and suitably broad definition. He conceived of it as the sum total of the forces of attraction between successive or simultaneous notes of a scale.[47] (Milne, 2013:7)

Milne (2013:7-8) takes a related definition, defining tonality as an organization of pitches that relates certain pitches or chords in terms of attraction to other pitches and chords. This organization is exhibited by pitches related simultaneously or in succession.

The affinities of points and movements within a relational context are defined by relative tensions. Such a general notion of tonality as that introduced here, may be further elaborated by the notions of a tonic centre and associated sets of tones forming tonal relations.

> The whole mass of tones and the connection of harmonies must stand in a close and always distinctly perceptible relationship to

[43] The term 'tone' is derived from the old Greek word '*pthongos*', meaning tension (Barker, 1989:266-269).

[44] A B C D E F G / La Ti Do Re Mi Fa So / Dha Ni Sa Re Ga Ma Pa / 6th(minor root), 7th, 1st(Major root), 2nd, 3rd, 4th, 5th

[45] *A*♭ *B*♭ *D*♭ *E*♭ *F*♯ / la ti re mi fa / dha ni re ga ma / flattened 6th, flattened 7th, flattened 2nd, flattened 3rd, sharpened 4th, flattened 6th

[46] A Bb B C Db D Eb E F *F*♯ G Ab / La ti Ti Do re Re mi Mi Fa fa So la / Dha ni Ni Sa re Re ga Ga Ma ma Pa dha / 6th(minor root), flattened 7th, 7th, 1st(Major root), flattened 2nd, 2nd, flattened 3rd, 3rd, 4th, sharpened 4th, 5th, flattened 6th, the harmonic chromatic scale (Prout, 1903:200).

[47] See Hyer (2001)

Stellenbosch University https://scholar.sun.ac.za

> some arbitrarily selected tonic, and the mass of tone which forms the whole composition must be developed from this tonic, and must finally return to it. (Helmholtz, 1895:249)

Dahlhaus (1980) and Hyer (2001) have investigated the origins and uses of the term tonality and relate this to a variety of meanings. It may be used in a broad sense as 'the systematic organization of pitch phenomena in both Western and non-Western music', or 'a rational and self-contained arrangement of musical phenomena' (Hyer, 2002:727).It may also take a narrower definition specifying some particular such arrangement as 'a system of relationships between pitches having a "tonic" or central pitch as its most important element' (Dahlhaus, 1980:52). This last definition of tonality refers to a specific systematic organization of pitches developed in the early seventeenth century, as contrasted with modality, and atonality. It may take a narrower definition, where it is used as a synonym for key. (Milne, 2013:7-8)

> There is also some ambiguity about whether tonality refers to the organization of pitches, to the music that results from this organization, or to the feelings these organized pitches induce. (Milne, 2013:7)

Tonality, as a relational structure forged by technical developments within the western canon of music, derives its relations by analogy from cognitive observations of sounds. Language itself may disguise assumptions that are being made. In music such linguistic assumptions may be commonly found in regarding pitch distribution, the invariance of intervals[48], and modulatory theory. The complexity underpinning these factors is obviated by the simple analyses of pitch practices that are generally used for instructing practical tuition.

[48]Intervallic invariance across pitch space is a cherished characteristic of equally-tempered intonations, however the geometric progression of these parts in physical space make certain compromises to the natural laws of sympathetic resonance upon which musical science was founded. Many musicians have rejected these compromises. (Helmholtz, 1895:325)

Stellenbosch University https://scholar.sun.ac.za

Chapter 3

Proposed Mathematical Modeling of Pitch Relations

We shall have missed the truth if we make that which judges neither the limit nor the ruling principle, and into the limit and ruling principle that which is judged.

–Aristoxenus (Barker, 1989:157)

3.1 Hierarchies of Means

HISTORIC models that regard the division of pitch space make use of various combination of mathematical *means*. The arithmetic and geometric means are particularly important in this regard, each giving rise to what may be broadly termed as *systems of relative invariance*[1]. The arithmetic and geometric means describe rational and irrational hierarchies of pitch relations. Proceeding to order these systems of pitch relations, as successive n-limits and EDO systems, introduces hierarchical similarities of these systems, amongst each means own members, and also in the cross relations of these hierarchies to each other.

Common to both systems is the consideration of a radix point, or some quantity taken as a unit, against which one may form a scale of relations. The number one, 1, defines a generalised unit. Proceeding from this the relation,

[1] Conversely these may also be described as systems of relative variance. The relativity of invariance and variance is determined by the variable definition of *invariance* seen according to each mean.

An invariantly relative arithmetic series of distances, such as 1, 2, 3, 4, 5, 6, ... , is at the same time geometrically variant. An invariantly relative geometric series of distances, such as 1, 2, 4, 8, 16, 32, ... , is similarly arithmetically variant.

Stellenbosch University https://scholar.sun.ac.za

presented as a ratio, but representing the inverse relations of vibration number and sounding length, the ratio of $^1/_1$, one to one, defines a generalised unity. This may be said to be the basic relation of isolated tones to themselves.

The rational hierarchy proposed here proceeds by simple arithmetic means, beginning with the number *one*, representing unity ($^1/_1$) in pitch relations, and the doubling ($^2/_1$) and halving ($^1/_2$) relations represented in the number *two*, this forming the mathematical basis of the octave interval. To these relations are added those formed in combination with the number *three*, the generative principle of the 3-limit Pythagorean interval relations of the form $^{3^n}/_{2^m}$ and $^{2^m}/_{3^n}$, where m and n are integers. Successive prime number generators contribute further variations through the various n-limits systems and their various multiplicative expansions and compounds.

The hierarchy governing the irrational relations of pitch, as defined by the various Equally Divided Octave (EDO) temperaments, applies geometric means to the arithmetic principle of the octave division. 2-EDO represents the 12-TET tritone, and is found in all even numbered EDO systems. 3-EDO represents the 12-TET augmented triad relations and is found in all EDO systems that are multiplies of three, such as 9-EDO, 27-EDO, or 33-EDO. 4-EDO represents the 12-TET diminished tetrad relations and is found in all EDO systems that are multiplies of four, such as 16-EDO, 28-EDO, or 54-EDO. Pursuing such further shows that many of the higher numbered EDOs contain only relatively few unique intervals that are not already encapsulated by a lower numbered EDO, with the prime numbered EDOs furnishing the greatest number of unique pitch identities.

3.1.1 Linear Representation

Simple linear representations of pitch distances, as one-dimensional representation, are complicated by the variable nature of the means used to express the relative pitch *field of distance*. Invariant distributions of linear scale may be either geometric or arithmetic. Within each, there are many possible graduations to their distributions. Some of these varieties are exhibited in the varieties of equally tempered logarithmic scales, and the various just intonations.

Shepard (1982:3) observes that there is a notable limitation stemming from the use of a one-dimensional framework being applied to the investigation of an innately manifold phenomena, such as the perceived pitch relations of a sequence of sounds.

> The failure of musically significant relations of pitch to emerge as invariant in these scales seems to be a direct consequence of the assiduous avoidance, by the psychoacoustic investigators, of any musical context or tonal framework within which the listeners might interpret the stimuli musically. (Shepard, 1982:3)

Stellenbosch University https://scholar.sun.ac.za

Uni-dimensional scalar relations result in numerous analytical ambivalences. It may be argued that these limitations are not necessarily problematic, as long as the focus of such study is not biased towards any individual scale of reference but accommodates these manifold perspectives by relating their compounded inter-relations. Such a view takes a thoroughly n-dimensional view of uni-dimensional relations.

Inter-relations of divergent scales of reference form hierarchies of pitch relations, with orders ranked according to their unique mathematical attributes. To further simplify these pitch relations their identities may be reduced to those of a single octave gamut. These pitch identities usefully describe gamuts of tone relations, while octave transpositions of these identities complete the full range of sounded pitch, and are useful for defining exact harmonic and melodic contexts.

3.1.2 Octave, Fifth and Fourth

> [Due to] the unidimensionality of scales of pitch [...], perceived similarity must decrease monotonically with increasing separation between tones on the scale. There is, therefore, no provision for the possibility that tones separated by a particularly significant musical interval, such as the octave, may be perceived as having more in common than tones separated by a somewhat smaller but musically less significant interval, such as the major seventh. [...] the [...] log-frequency scale, [...] being unidimensional, is subject to this [...] limitation. Yet, in the case of the octave, which corresponds to an approximately two-to-one ratio of frequencies, the phenomenon of augmented perceptual similarity at that particular interval [...] has long been anticipated [,] in fact empirically observed[*... and*] probably underlies the remarkable precision and cross-cultural consistency with which listeners are able to adjust a variable tone so that it stands in an octave relation to a given fixed tone." (Shepard, 1982:3)

The octave, as a basic musical phenomena exhibiting simple mathematical relations, is an important basis for the analysis of mathematical pitch relations, and serves well for the theoretical simplifications of analysis.

The number two, 2, defines a doubling of the unit, $^2/_1$. Conversely this number also represents a unit divided into two equal parts, $^1/_2$. The multiplication of two by itself yields a series of numbers other than itself. Namely, four, or 2^2; eight, or 2^3; sixteen, or 2^4; thirty two, or 2^5; sixty four, or 2^6; on hundred and twenty eight, or 2^7, and so on. These numbers are related to each other as octaves, as doubling or halving of sounding lengths yielding a series of notes that stand in octave relation. Thus, for example, a double octave may be represented as $^4/_1$, a triple octave as $^8/_1$, and so on.

Stellenbosch University https://scholar.sun.ac.za

This is the most pronounced pitch relation in music, and forms the foundation of interval recognition. No discussion of temperament or tuning may proceed without noting that in doubling, or halving, any unit measurement another unit measure is revealed that resembles the original in a very unique and important way.

> The resemblance of an Octave to its root is so great and striking that the dullest ear perceives it; the Octave seems to be almost a pure repetition of the root, as it, in fact, merely repeats a part of the compound tone of the root, without adding anything new. (Helmholtz, 1895:369-370)

The octave may be identified as perhaps the most intrinsic distinction of pitch, widely recognised since antiquity as representing a musical interval having "the greatest degree of simplicity, equality and stability."(Barker, 1989:375) The simple relations of doubling and halving which are represented by the octave are used to ground most systems of musical relations. The octave is widely recognised as the most consonant consonance

Many schemes of temperament acknowledge the immutability of the *pure* octave quantity $^2/_1$, and preserve this proportion alone as a distinctly arithmetic measure. Divisions of equal temperament, such as 12-TET, are applied to an already assumed arithmetic division of pitch, the octave $^2/_1$, hence the monicker of Equally Divided Octave, or EDO, temperaments.

There are some notable scales forming exceptions to the rule of the octave, such as the α, β and γ scales of Carlos (1987) and the Bohlen-Pierce[2] temperament and notation, with their use of the primary division of a *tritave*, or twelfth, of $^3/_1$. While these instances are exceptional, the notion of such *tritave* temperament that these instances indicate may prove useful to future development from this initial research. For the moment this research will deal only with temperaments based upon the simplest possible consonance of the octave.

> [...] the addition of any concordant interval to the octave makes the magnitude resulting from them concordant. This is a quality intrinsic and peculiar to the concord of the octave. [...] This is not the case with the first concords [the fourth $^4/_3$ and fifth $^3/_2$]. (Barker, 1989:160)

Within the range of the octave, there are two further commonly recognised consonances, the fourth of $^4/_3$ and the fifth of $^3/_2$. Both of these intervals, notionally due to their mathematical simplicity, are recognised as powerful harmonic relations. The fifth, according to its greater simplicity, is ranked higher and accorded a position of dominance in conventional harmonic theory.

---

[2] See Mathews *et al.* (1988); Mathews *&* Pierce (1989).

Stellenbosch University https://scholar.sun.ac.za

The number three, 3, defines a tripling of the unit, $^3/_1$. Similarly it may represents a measure that is one and a half again as great as a doubled unit, $^3/_2$. This last relation is very important to musical relations, forming the basis of the *Pythagorean* tuning system. Three may also be multiplied by itself yielding the relative series: 9, or 3^2; 27, or 3^3; 81, or 3^4; 243, or 3^5; 729, or 3^6, and so on.

> There are, indeed, a number of converging reasons for supposing that the fifth should, like the octave, have a unique status. (Shepard, 1982:8) [3]

The fourth, was accorded a high place of its own in the tetrachordal basis of musical melody and scale formation. It is indicative as it is of the practical inversion of arithmetic harmonic means, that the fourth and fifth added to each other constitute an exact octave. Similarly, if a fifth is subtracted from an octave it yields a fourth, and a fourth subtracted from an octave yields a fifth. Further, in the same manner, the fourth subtracted from the fifth, yields the epogdoic tone of $^9/_8$, and so on, this basic practice of fitting intervals together forming the basis of the *'harmosmenon'*, attunement or fitting together, of the classical Greek harmonic theories.

Pythagorean intonational theory, named after the apocryphal Greek harmonic teacher of the same name, recognises the fifth and the fourth as inversions of an intrinsic divisive relation and attempts to preserve this interval in all its proportions, hence the 'large' third of $^{81}/_{64}$, derived not as a harmonic third, but as the fifth of the pythagorean sixth of $^{27}/_{16}$, itself the fifth of the epogdoic tone. Pythagorean cycling of fifths, and the notion of their dominance over other intervals within the octave, has a unique position of pre-eminence in western canon. Notably, the 12-TET circle of fifths forms a device for ordering cadences and key relations.

Fifths and fourths, as integral units in Pythagorean and other 'just' intonation schemes, have quite different functions to octave divisions. While cycles of octaves reproduce only limited sets of doubling and halving pitch relations, cycles of fifths and fourths produce steadily divergent sets of *related* pitches, used in Pythagorean intonation to describe all aspects of tonal relations. These

[3] "Despite these diverse indications of the importance of the perfect fifth, the fifth has largely failed to reveal its unique status in psychoacoustic investigations for the same reason, I believe, that the octave often revealed its unique status only weakly, if at all. In the absence of a musical context, tones - particularly the pure sinusoidal tones favored by psychoacousticians - tend to be interpreted primarily with respect to the single, rectilinear dimension of pitch height. Without a musical context there is insufficient support for the internal representation of more complex components of pitch-components that may underlie the recognition of special musical intervals and that (like the chroma circle) are necessarily circular be- cause, again, the musical requirement of invariance under transposition entails that each such component repeat cyclically through successive octaves."(Shepard, 1982:8)

Stellenbosch University https://scholar.sun.ac.za

relations may be expanded to as many steps as one desires without ever cycling back upon the exact octave analogy of the starting pitch. Methods of musical temperament consider this to be a chief deficiency of such 'pure' intonations, such intrinsic divergence necessarily arising from the consideration of intervals created from relations of integral numbers (Ellis, 1863*b*:404-422). Prime numbers relations, as prime governing forces behind most just and pure intonations, constitute the various n-limits.

3-limit intervals form the basis of what is commonly termed *Pythagorean* tuning. This divergent, non-cyclic, system may be extended by as many cycles as is required. 7 cycles of a 3-limit generator ($^3/_2$ or $^4/_3$), taking their pitches in a central octave, suitably defines a single tonal region of seven diatonic notes, while 12 cycles adequately fleshes out a chromatic scale and allows for a limited tonal modulation, with certain harmonic voicing being necessarily restricted. The extension of this system to 19, 22, or 24 cycles, allows for the description of adjusting chromatic regions, reducing the limitations available to tonal modulations and harmonic voicing and introducing various enharmonic identities.

Unique 5-limit intervals are defined in this research as *Ptolemaic* tuning schemes, and constitute additions, or alterations, to the 3-limit intervals. Similarly, unique 7-limit intervals mark *Septimal* tuning schemes. One may continue this practice infinitely, taking the various higher prime numbered limits as additions, or alterations, to the above sets. These three above-mentioned sets are notable for constituting the prime integer limits.

The considerations of arithmetic methods of intonation acknowledge the respective intervals of these n-limit tunings as varieties of enharmonically related intervals, these share functions but have unique mathematical identities and pitch relations.

3.1.3 Musical Circles

Musical circles have long been popular devices in the teaching of musical theory.

Such an approach is indicated by Plato when he describes a string divided in half, each half joined to it's own ends, thus forming two circles each representing an octave[4]. Each circle contains the intervals from unity to the octave, as contained in the first half of the string, arranged along its circumference (Barker, 1989:60).

These arrangements of intervals are posited upon the simple representation of a cyclic phenomena as a circle, representing the octave phenomena of pitch duplication in a simplified form as a single point upon the circumference. The circumference may be divided by geometric chords, lines, and tangents, these describing the relations of points.

[4] See subsection 2.2.2

Stellenbosch University https://scholar.sun.ac.za

Musically useful circular arrangements are derived from the simple notion that the circular design represents the similarity of notes within a generalised octave, and the rules generally applicable to one octave may be transposed to any specific octave span.

Musical circles represent, at all points/pitches along their linear continuum, both the beginning and the middle of a theoretical string tuned to that pitch, taking the beginning of the string to represent the interval given by the original length of the string as $^1/_1$ and the middle to represent the half length of unity, the octave $^2/_1$. Thus displayed, intervals distant by an octave are found upon the same point along the circles circumference, this kind of view is most convenient for data dealing with generalised pitch usages or condensations of more complex multi-octave pitch data.

Such musical circles may usefully be plotted either arithmetically[5], or logarithmically.

Logarithmic plotting results in similar intervals having the same lengths regardless of their position around the circle. This makes logarithmic plotting of the octave of great use to theoretical discussions of intervals, and accordingly makes it the most popular form of the musical representation of pitch by such circular display. The resulting logarithmically divided circle displays a greater degree of homogeneity of intervals and is crucial to popular representations of such devices as the *12-TET circle of fifths* and the related *12-TET chromatic circle*. However, the arithmetic method is to be greatly esteemed for practical instruction, as:

> To divide a sounding string [*or any other sounding body*] into equal parts is not equivalent to a division of 'tonal' space into equal intervals. [...] Aristoxenus represented equal intervals as equal tonal 'distances', expressed in terms of tones and their fractions. (Barker, 1989:345)

The two-dimensional circular representation may also be expanded into a three-dimensional spiral to include the unused half of the string length, the intervals from the octave upwards.

Shepard (1982) describes the advantages of spherical helical representations of musical pitch, showing consistently placed octave equivalences (Shepard, 1982:4), and describes methods of realising such arrangement (Shepard, 1982:5). Shepard (1982:6) argues for a geometrically regular structure of pitch

[5] Such a divisible, and extensible, musical continuum of pitch is theoretically similar to the practices of the divisions of a string length upon a monochord. Taking a string length as $^1/_1$ (Root, generator or *unity*) and dividing the length in half effectively results in the simplest interval of an octave ($^2/_1$). Transferred to a diagram, with circular octave equivalences, this arithmetic scaling furnishes a useful musical pitch circle for the instruction of string division. In this representation similar interval relations display different lengths in different ranges around the circular octave plot.

Stellenbosch University https://scholar.sun.ac.za

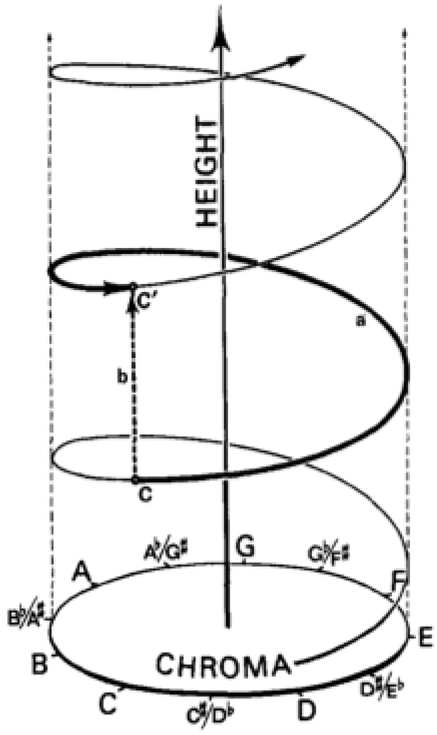

Figure 3.1: Illustration of Shepard's spiralling tonal arrangement (Shepard, 1982:308).

relations, suggesting "that if it is musical pitch that interests us, the representation should reflect the deeper structure that underlies a listener's competence to impose a musical interpretation on a stream of acoustic inputs under favorable conditions."[6]

Shepard (1982:14-23) expands his helical analogy into double helical models, with variations on these helical patterns obtained by cutting parts of the pattern in order to wrap a single transposable octave of chroma relations into a closed toroid. This single octave, wrapped on the skin of a toroidal topography, may in turn be extrapolated outward, again by cutting it, to form a helically spiralling cylindrical graph.

Such complications may be useful to descriptions of extended gamuts and highlight some important relations, however they do not represent an impor-

[6] (Shepard, 1982:8) goes on to describe how the circular octave representation may be expanded into a helical model "[...], however, the helical structure does not provide either augmented proximity or collinearity for tones separated by any other special musical interval."

" [...], beginning with Ebbinghaus, Drobisch's proposal of a helical representation has been criticized for its failure to account for the special status of the interval of a perfect fifth (See Ruckmick (1929).)." (Shepard, 1982:8)

Stellenbosch University https://scholar.sun.ac.za

tant distinction between the harmonic melodic contexts of pitch relations, their relations over time.

3.1.4 Notation

The adjustable pitch available to a freely-pitching voice or instrument, and certain theoretical complications and performance fundaments unique to unfixed pitch practices, are limited by the use of buttons or software functions providing systematically graded pitch step sizes. Such distinctions between approaches to theoretical systems of pitch are important, as the reduction of pitch analysis to a set of parameters may be analytically useful when understood as being practically important to musical performance. Distinctions of fixed and unfixed pitch structures mark topographically variant perspectives of the ordering of a common phenomena.

The notational methods proposed by Ellis[7], based upon the work of Helmholtz and Hauptmann, are useful for the purposes of scientific investigation, forming a simple bridge between the mathematics of intonation and the notation of western music.

> [...] Western [...] notation is in fact based on the triple septenary of seven naturals [♮], seven sharps [♯], and seven flats [♭]. If the octave is added, this gives 3 x 7 + 1 = 22. (Danielou, 1995:82)

The logarithmic means of frequency numbers, the standard scientific method for presenting the computation of audio into musical data, allow for the plotting of a usefully invariant scaling. Frequency numbers in hertz may be translated to cent relations, these in turn analysed by correlation to dynamically located maps of harmonically invariant tonal structures expressed in cent relation to a dominant pitch as tonic. Tonal structures may be computationally determined by correlation with further information regarding pitch intensity and recurrence.

This correlation of a rational and dynamic tonal modeling upon an invariant scale of frequency relations produces a musically descriptive model regarding pitch ratios and exact cents deviations. This may be simplified into traditional nomenclature, with some additions such as those proposed by Helmholtz-Ellis, after the model first investigated by Hauptmann (Helmholtz, 1895:276-279, 315).

In classical western music notation letters, called *notes*, represent both certain *tones*, or intervals, and their *pitch* or frequency[8]. Helmholtz-Ellis notation proposes a definite mathematical premise underlying the first seven letters of the alphabet, rendering A, B, C, D, E, F, and G, so that:

---

[7] See figure 3.2.

[8] The number of double vibrations contained in one second.

Stellenbosch University https://scholar.sun.ac.za

1. *Following a Note.*

♯ $= \frac{135}{128} =$ sharp, or greater limma.

♭ $= \frac{128}{135} =$ flat, or hypolimma.

× = ♯.♯ = double sharp; ♭♭ = ♭.♭ = double flat.

2. *Preceding a Note.*

† $= \frac{81}{80} =$ acute, or comma.

‡ $= \frac{80}{81} =$ grave, or hypocomma.

ʐ $= \frac{63}{64} =$ septime (an inverted 2).

¶ $= \frac{32805}{32768} =$ 1·001129150390625 = schisma.

‖♭ $= \frac{32768}{32805} =$ 0·99887212315 = hyposchisma.

The name and pitch of the tones represented by any such notes as

‡*c*♯ = grave *c* sharp $= \frac{80}{81} \cdot c \cdot \frac{135}{128} = \frac{25}{24}\, c$,

†*e*♭ = acute *e* flat $= \frac{81}{80} \cdot e \cdot \frac{128}{135} = \frac{24}{25}\, e$,

and the ratio of their pitches to the corresponding notes in the scale of *C* major is therefore precisely indicated. In ordinary musical notation on the staff, it is only necessary to prefix the signs †, ‡, ʐ, ¶, ‖♭, to those already in use. These symbols suffice for writing any tone whose index is the product $2^m . 3^n . 5^p . 7^q$ For equally tempered tones, when it is necessary to distinguish them, the sign ‖ is prefixed to the usual names, and read "equal." Since

‖*g* : *c* $= \sqrt[12]{2^7} : 1 =$ 0·998871384584 × $\frac{3}{2}$,

and

‖♭*g* : *c* = 0·99887212315 × $\frac{3}{2}$,

we may without sensible error consider ‖*g* = ‖♭*g*, and hence represent the equally tempered scale

c, ‖*d*♭, ‖*d*, ‖*e*♭, ‖*e*, ‖*f*, ‖*f*♯, ‖*g*, ‖*a*♭, ‖*a*, ‖*b*♭, ‖*b* by

c, ¶⁵‡*d*♭, ‖♭²*d*, ¶³*e*♭, ‖♭⁴†*e*, ¶*f*, ‖♭⁶†*f*♯, ‖♭*g*, ¶⁴*a*♭, ‖♭³†*a*, ¶²*b*♭, ‖♭⁵†*b*.

Figure 3.2: Helmholtz-Ellis (HE) Notation

> $8D = 9C, 4E = 5C, 3F = 4C, 2G = 3C, 3A = 5C, 8B = 15C$ [; and,]
> $c = \frac{1}{2}, c^2 = \frac{1}{4}, c^4 = \frac{1}{8}$, [...] , $2C = 4, C^4 = 8$, [...]
> and similarly for other letters. (Ellis, 1863*a*:94)

Further extensions in the form of symbols for the various comma and dieses discrepancies complete this system of notation. These extensions to standard musical notation form undeniably useful additions for those interested in microtonal composition, however, the practical utilisation of such scored notation relies upon many conventions that have yet to become ingrained in common musical instruction. Not least of which is that of adopting Pythagorean temperament as the standard for unannotated pitch relations. The utilities of the dieses and the commas are of great importance for virtuosic practice, their recognition being a key point upon which this extended notation is based.

Stellenbosch University https://scholar.sun.ac.za

It is interesting to consider some possibilities for the further extension of such a notation to describe timbral aspects of of sound. Timbral approaches to *tone-colour* descriptions have been explored by researchers such as Lerdahl (1987) and Sethares (2005), with the result that timbre has come to be represented in the main by complex multi-dimensional structures, related to more commonplace pitch structures but differing in many important ways. While pitch and timbre share a common unit, the sound, they differ in their dimensionality. Pitches, as musical tones, may be represented in the horizontal plane of movement by the unit lengths of rhythmic phrases and metric repetitive structures, and in the vertical plane relative unit height. These dimensions may be further superimposed, as on a musical stave or other two dimensional diagram, for instruction and analysis. Timbre however shows itself in a more remote dimension, that of the spectrum of each sound at each point in time. A systematised compound representation of each sound occurrence, as both a discreet fundamental pitch and its partial overtone components, relates these to a fundamental reference pitch.

Experiments involving a single sustained drone sound, changing over time only in the strength of overtone reinforcement[9] (Lerdahl, 1987:150-155), show evidence to suggest that timbre is an important aspect of tone production and melodic and harmonic musical practices. The lack of formal systematising of timbre relations, such as has long been furnished to the more obvious musical aspects of fundamental pitch and duration relations of sounds, is in part due to the inherent difficulty of negotiating the representation of a further dimension of musical measurement and description (timbral relations) in a practical utilitarian way for musical ends.

While computerised methods of tone production and synthesis have allowed for the development of some interesting timbral control utilities, from rotary pitch shift controllers to command-line software arguments, these systems tend to lack the intuitive potential of instruments without fixed pitches[10] such as voices and violins. Timbral changes are effected by fixed-pitch instruments as settings, but by freely-pitching instruments as interpretative functions, arising from nuances of tone production. The critical difference between these approaches lies in the practical separation between secondary spectral modifiers, such as constitute the parameters of synthesis, as opposed to the modification of timbre by the method of performance itself.

[9] These modelled as vowel sounds using the CHANT synthesis program by Rodet *et al.* (1984).

[10] That is, the limits of a voices range tend to be fixed, just as a string may not produce a sound below a tuned 'fixed' pitch. However, the potential ranging of this 'fixture' may be placed at any point along the continuity of the pitch spectrum. A string may be tuned to any sound along the spectrum of sound, any sound its material strengths may allow for or be modified to allow, and voices are not discreetly ranging, but rather show themselves to bear overlapping ranges.

Stellenbosch University https://scholar.sun.ac.za

3.2 Modelling Historic Pitch Theory

In order to approach the task of defining a methodology for modeling the musical complexity surrounding pitch analysis, this research will suggest that some interpretative difficulties, evident in the limitations imposed on and by software tools in the attempt to return pitch relations as musically defined pitch interval classes, may be overcome by a rigourously defined mathematical approach to the methodology of pitch classification and analysis. Software definitions of musical pitch distinctions have yet to reach common consensus regarding pitch complexity. A formal consensus capable of regarding these complexities in terms of mathematical statement would greatly assist the succinct and simple, yet explicitly accurate, rendering of musical information regarding pitch data.

3.2.1 Defining Intervals Mathematically

The musical theory of the western classical tradition offers many varying views on intervallic sonance. Historic pitch practices exhibit a great variety of acceptable temperaments and tuning systems. These may be classified according to the mathematical relations of their intervals.

> From a purely theoretical point of view there is no least interval.(Barker, 1989:160)

Equal temperaments of musical pitch are based upon the desirability of some small incomposite interval, or ‘atomic’ part, out of which all other desired composite intervals may be constructed. The 12-tet semitone answers generally to such a purpose. Equal-temperament arrangements of pitch are composed from homogeneous intervals, all relatable as composites of a single size of smallest step. Rational tunings, however, tend to result in a variety of small steps.

Mathematically, equally-tempered musical relations may be seen as being formed from larger divisions which when arranged give the impression of overlapping to form equal divisions of their parts. This view also allows for their relation to schemes of unequally sized divisions, these formed from n-limit generators whose overlapping does not form equal divisions of their parts.

> [...] there are eight magnitudes of concords[11]. The smallest is the fourth [$^4/_3$]: that is the smallest is determined by the nature of melody itself. This is shown by the fact that we sing many

[11] Peculiarly, after telling us that there are eight magnitudes of concord, Aristoxenus enumerates only three directly. Presumably, the remaining five concords are duplications of those preceding, i.e. the eleventh $^8/_3$, twelfth $^3/_1$, Double Octave $^4/_1$ (15$^{\text{ve}}$), eighteenth $^{16}/_3$, and nineteenth $^6/_1$.

Stellenbosch University https://scholar.sun.ac.za

> intervals smaller than the fourth, but all of them are discordant. The second is the fifth [3/2]: every magnitude which there may be between these two will be discordant. The third is the octave [2/1] all magnitudes between the fifth and the octave being discordant. (Barker, 1989:160)

The epogdoic tone was described by the difference between the two smallest concords.[12]

The epogdoic tone, with the addition of another such tone of similar, though not necessarily the exactly same, size constituted the various ditones, which as the name suggests were composed of and divided into two similar intervals. To the ditone a hemitone was added to produce the interval of the fourth. Conversely a hemitone removed from a fourth gave a ditone, and a ditone removed from a fourth gives a hemitone. [13]

The *hemitones* of classical Greek were related intervals, defined by variously sized intervals within a certain range[14].

Euler[15] represented 5-limit pitch relations by the formula: $2^m.3^n.5^p$. From this formula, Euler formed different "genera musica" by varying n and p from 0 to fixed limits. The practical application of this method may be exhibited by what Euler called his "genus diatonicum hodiernum", which sets a limit of 3 to n and 2 to p, and consists of 12 tones. He further makes distinctions between the interval forms of 12 tone scale that may be achieved by such a method, giving a scheme of two keyboard manuals each of which gives a different set of the 12 tones, thus making for easy comparisons. (Ellis, 1863*a*:93, 102-3)

Similar methods of comparison were used by Helmholtz, who would often compare justly tuned manuals with others tuned to Pythagorean and Equal temperaments (Helmholtz, 1895:316-319). The purposes of these experiments were to investigate the practicalities of such intonation schemes and assess their relations. It was asserted by Helmholtz that the scheme of relations pursued by the Greek harmonic theories were correctly based in musically important physical phenomena and that these relations form the basis of a holistic approach to harmonic relations, having as its only drawback the immensity of development required to construct a truly all-inclusive framework (Helmholtz, 1895:234-371).

[12]"The [*epogdoic*] tone[9/8] is that by which the fifth [3/2] is greater than the fourth[4/3]." (Barker, 1989:160)

[13]"[...]the fourth is two and half tones." (Barker, 1989:160)
"[...] the fourth [4/3] is two and a half tones.("[...] or thirty twelfths of a tone)" (Barker, 1989:345)

[14]The largest being Ptolemy's *even* semitone of 12/11 (150 cents) and the smallest being the lesser chromatic semitone of 25/24 (70 cents). (Barker, 1989:350)

[15]Leonhard Euler, the swiss mathematician(1707-1783). For more regarding his life and mathematical work see Bradley *et al.* (2008).

Stellenbosch University https://scholar.sun.ac.za

From this complexity we may construct various useful hierarchies of relations ordering the mathematical ranges of musical pitch intervals by proceeding to investigate the limitations imposed by classical Greek harmonic science upon the pedagogy of pitch relation.

3.2.2 Tetrachords, Genera and Laws of Melodic Succession

> [...] it is impossible for differences in the magnitudes of intervals always to follow upon differences in notes. [...] the converse relation does not hold [...] (Barker, 1989:161)

Classical Greek theory treated the interval space of the fourth as being a particularly useful space, as the intervals within it's span could be grasped at once by the four fingers of each hand. Their tetrachord systems use the same names to indicate fingering and to define note positions[16]. They further perceived that varieties of musical relations could be simplified into disjointed and conjoined sets of tetrachord spans and their recurrences. Thus viewed, a major diatonic scale is seen as a single repeating tetrachord[17] ascending first disjunct, then conjunct. This model is useful to fretless string players, in that it represents very accurately the shift that must needs take place after the fourth finger in sequence has ascended upon a single string in a single position. Whether horizontally, along the same string, or vertically, to an adjoining string, the fingers of the hand must shift to encompass any span greater than the fourth[18].

Just as the relations of the many possible octave positions may be simplified by a consideration of a single generalised octave, so the numerous interval compounds possible within the octave may be usefully treated as being intrinsically composed of smaller tetrachord units.

> Melodies fall into three genera, the diatonic, the chromatic and the enharmonic [...] every melody is either diatonic or chromatic or enharmonic, or a mixture of these, or common to them. (Barker, 1989:159)

[16]See the usage of the terms *hypate, parhypate, lichanos* and *mese* in the description of the division of the lesser and greater perfect systema (Barker, 1989:266-269).

[17]Generally speaking, this particular major diatonic tetrachord goes by the formula of *tone, tone, semitone*, ascending.

[18]Or the raised fourth, though extesnions of fingering may extend this range, they are exceptional and rarely exceed the range of a fifth. These extensions are provided for in classical Greek tetrachord theory by the notion of *Proslambonomenos*, or 'that number which adjoins the ranges boundary'. For more regarding this see Barker (1989).

Stellenbosch University https://scholar.sun.ac.za

> Of the parts of the tone the following are melodic: the half, which is called the semitone, the third part, which is called the least chromatic diesis, and the quarter, which is called the least enharmonic diesis. (Barker, 1989:161)

Generally stated, the diatonic genera of tetrachords are composed of two tones and a semitone; the chromatic genera are composed of a semiditone(minor 3rd) and two semitones; and the enharmonic genera is composed of a ditone(major 3rd) and a *pyknos*/compression, consisting of two quartertones (Barker, 1989:143, 146, 160).

These last definitions are generalised statements of the broader characteristics of these interval arrangements, and only generally takes into account the wide ranging mathematical variations which occur in their arrangements in context.

> A person who sets out signs to indicate intervals does not use a special sign for each of the distinctions which exist among intervals - for instance, for the several divisions of the fourth produced by the differences between the genera, or for the several arrangements produced by alteration in the order of combination of the incomposite interval. (Barker, 1989:156)

The classical Greek harmonic sciences derived much of their material from the pursuit of these perceptions regarding the relations of sounds. Proceeding from these are the rules laid out by Aristoxenus in his laws of melodic succession, arising from his study of the genera and tetrachords formed from and forming musical intervals (Barker, 1989:170-183). These laws provide some directions for the usages of the smaller intervals within the spaces of the fourth and fifth, and also for the successions of tones, hemitones and ditones in ascent and descent. Furthermore, Aristoxenus makes special mention of divisions of the *pyknos*, compressions, dividing the hemitones into two unequal ranges, bounded by three points (Barker, 1989:181).

> If we seek individual names corresponding to every increase and decrease in the interval of the pyknon, we shall evidently need an unlimited number of names. (Barker, 1989:162)

3.2.2.1 Non-Melodic Successions

Directions contained in the writings of the classical Greek harmonists give instances of non-melodic pitch successions, described as being critical to the performance of certain incarnations of the chromatic and enharmonic genera. Such critical enharmonic differentiation does not take the form of very small intervals in continuous succession, but of small interval variations and compensations amongst larger interval successions. The former arrangement, though functionless, was termed 'katapyknosis' by the terms of classical Greek theory.

Stellenbosch University https://scholar.sun.ac.za

> It will become clear in the course of our investigation that *kata-pyknosis* is unmelodic and altogether useless.(Barker, 1989:154)

The functional uses of non-melodic successions take the form of instances of deviations, adjustments, and accents, of pitch, often described in terms of expressive pitch and grace ornamentation.

> [...] We should not forget that the problem is not to play intervals of one comma in succession but to play intervals with an accuracy of one comma. A difference of one comma in a fifth or an octave is not only perceptible but extremely disagreeable even to an untrained ear. The same difference in a third or in a major second (it is then the difference between the major and the minor tone) completely changes the color of the note and its expression. One can even say, as a rule, that such differences are the very basis of vocal and melodic expression [...] (Danielou, 1995:55)

The classical Greek pyknon share commononalities with the Indian Classical Music (ICM) notion of *sruti*, a very broadly defined necessity of ICM pitch practice.

> The sruti is not a tempered interval. Not all srutis are equal [...] Like every practical division of the octave, the division of the srutis is theoretically insufficient [...]
>
> The term sruti, like the Greek quarter tone, refers to the displacement of a note by an interval smaller than a half tone in order to obtain a harmonically accurate interval. (Danielou, 1995:81-2)

These distinctions arise mathematically from considerations of the intervals of the various n-limits. 3-limit relations produce schismas, 5-limit intervals produce commas and dieses, and septimal relations produce further varieties of commas and dieses. These are exhibited in the great variety of enharmonic intervals differing by these small intervals.

3.3 Tuning and Temperaments

Viewed dynamically, temperaments and tuning systems may be compared as systems that create and cancel enharmonic equivalences (Mandelbaum, 1961:xiv).

Helmholtz argued that the disadvantages of tempered tuning renders them inferior to just intonations (Helmholtz, 1895:322-326). Having conclusively established the physical existence of differential and summational combinational tones, Helmholtz (1895:152-159) makes these combinational tones the basis for his justification of the musical superiority of systems of just and Pythagorean

Stellenbosch University https://scholar.sun.ac.za

intonations, together with their various commas and other discrepancies, over methods of temperament that average out such complex pitch discrepancies for the conveniences of fixed pitch instrument.

Ellis however takes a very different approach, arguing that such distinctions as regards intervals that come to differ by the small value of the syntonic comma ($^{81}/_{80}$) should be abolished, "on account of the number of fresh relations that would be thus introduced." (Ellis, 1863*b*:404)

> ... the musical scale which introduces the comma consists of tones whose pitch is formed from the numbers 1, 3, 5, by multiplying continually by 2, 3, and 5. Hence to abolish the comma it will be necessary to use other numbers in place of these. But this alteration will necessarily change the physical constitution of musical chords, which will now become approximate, instead of exact representatives of qualities of tone with a precisely defined root. It is also evident that all the conjunct harmonics will be thus rendered pulsative, and that therefore all the concords will be decidedly dissonant at all available pitches. (Ellis, 1863*b*:404)

The pulsative beating that accompanies the sounding of tempered intervals is due to the nature of the differential and summational combinational tones produced by tones which are not related by the integral quantities of whole number relations. "This is an evil which cannot be avoided by any system of temperament, and is about equally objectionable in all systems. It may therefore be also left out of consideration in selecting a temperament." (Ellis, 1863*b*:405)

Within tempered systems it is convenient to refer to the smallest *units*, due to the ambiguities of the terminology of *semitones* in tuning systems having more than 12 tones (Mandelbaum, 1961:xiv).

3.3.1 Equal Temperaments

Mathematically, one may draw many fine distinctions between varying grades of pitch relations as intervals expressed as a ratio of magnitudes, and also in terms of rational and logarithmic deviations from simple ratios. There is often no way to convey such discriminating detail by a description of pitch and intervals such as they stand in modern musical pitch theory and common notation.

The practices recommended by Ellis (1863*a*:95-6) when considering relations involving temperament involve the calculation of their representative irrational intervals, represented mathematically as square root functions, as

Stellenbosch University https://scholar.sun.ac.za

interval ratios	cents	just interval	cents	error
$2^{0/12} = 1$	0	$^1/_1$	0	0
$2^{1/12} = \sqrt[12]{2}$	100	$^{16}/_{15}$	111.73	-11.73
$2^{2/12} = \sqrt[6]{2}$	200	$^9/_8$	203.91	-3.91
$2^{3/12} = \sqrt[4]{2}$	300	$^6/_5$	315.64	-15.64
$2^{4/12} = \sqrt[3]{2}$	400	$^5/_4$	386.31	+13.69
$2^{5/12} = \sqrt[12]{32}$	500	$^5/_4$	498.04	+1.96
$2^{6/12} = \sqrt{2}$	600	$^7/_5$	582.51	+17.49
$2^{7/12} = \sqrt[12]{128}$	700	$^3/_2$	701.96	-1.96
$2^{8/12} = \sqrt[3]{4}$	800	$^8/_5$	813.69	-13.69
$2^{9/12} = \sqrt[4]{8}$	900	$^5/_3$	884.36	+15.64
$2^{10/12} = \sqrt[6]{32}$	1000	$^{16}/_9$	996.09	+3.91
$2^{11/12} = \sqrt[12]{2048}$	1100	$^{15}/_8$	1088.27	+11.73
$2^{12/12} = 2$	1200	$^2/_1$	1200	0

Table 3.1: Some 12-TET and Just-Intonation Comparisons

logarithms to five decimal places[19]. By this process the intervals of rational numbered tuning systems and irrationally represented temperament schemes may be related by logarithmic cents as shown in table 3.1.

Equal divisions of intervals other than the octave result in further irrational mathematical identities following the same square root representation. This is shown in the smallest step of Bohlen-Pierce's[20] tritave (1 : 3) divisions into 13 parts, represented as $\sqrt[13]{3}$ (146.3 cents), and the smallest steps of the tuning schemes defined by Carlos (1987) as Alpha[21], Beta[22], and Gamma[23]

3.3.2 Just Intonations

> This relation of whole numbers to musical consonances was from all time looked upon as a wonderful mystery of deep significance. The Pythagoreans themselves made use of it in their speculations on the harmony of the spheres. From that time it remained partly the goal and partly the starting point of the strangest and most

[19] "In calculating relative pitches or intervals, and in all questions of temperament, it is most convenient to use *ordinary* logarithms to five places, because the actual pitches, and the length of the monochord (which is the reciprocal of the relative pitch), can be thus most easily found." (Ellis, 1863*a*:95-6)

[20] See Mathews & Pierce (1989).

[21] (α): $\sqrt[9]{3/2} = 78$ cents

[22] (β): $\sqrt[11]{3/2} = 63.8$ cents

[23] (γ): $\sqrt[20]{3/2} = 35.1$ cents

Stellenbosch University https://scholar.sun.ac.za

comma grave	cents	3-limit	cents	comma acute	cents
$^{160}/_{81}$	1178.5	$^{1}/_{1}$	0	$^{81}/_{80}$	21.5
		$^{256}/_{243}$	90	$^{16}/_{15}$	111.73
$^{10}/_{9}$	182	$^{9}/_{8}$	203.91		
		$^{32}/_{27}$	294	$^{6}/_{5}$	315.64
$^{100}/_{81}$	364	$^{5}/_{4}$	386.31	$^{81}/_{64}$	408
		$^{4}/_{3}$	498	$^{27}/_{20}$	520
		$^{1024}/_{729}$	588	$^{64}/_{45}$	610
$^{45}/_{32}$	590	$^{729}/_{512}$	612		
$^{40}/_{27}$	680	$^{3}/_{2}$	702		
$^{128}/_{81}$	792	$^{8}/_{5}$	813.69	$^{81}/_{50}$	836
$^{5}/_{3}$	884.36	$^{27}/_{16}$	906		
		$^{16}/_{9}$	996.09	$^{9}/_{5}$	1018
$^{15}/_{8}$	1088.27	$^{243}/_{128}$	1110		
$^{2}/_{1}$	1200	$^{2}/_{1}$	1200	$^{2}/_{1}$	1200

Table 3.2: Comma relations of 3-limit and 5-limit JI

> venturesome, fantastic or philosophic combinations, till in modern times the majority of investigators adopted the notion accepted by Euler himself, that the human mind had a peculiar pleasure in simple ratios, because it could better understand them and comprehend their bearings. But it remained uninvestigated how the mind of a listener not versed in physics, who perhaps was not even aware that musical tones depended on periodical vibrations, contrived to recognise and compare these ratios of the pitch numbers. (Helmholtz, 1895:15)

The cent relations proposed by Ellis (1863*a*:95-6) may be usefully used to compare different sets of intonation and temperament. Some of these differences are shown in table 3.1 and table 3.2.

Important distinctions of the syntonic comma ($^{81}/_{80}$) and the lesser diesis ($^{128}/_{125}$) arise from 5-limit intervals and their relations with the preceding 3-limit intervals. '*Septimal*' intervals of 7-limit systems further compound the relations of 3- and 5-limit systems, generating many hundreds of ratios that may usefully describe pitch movement in any certain context. 11-limit and limit systems are not uncommon, however the level of complexity which they introduce may be usefully modelled in further research using similar methods of compounding interval sets.

In order to diminish any loss of accuracy regarding pitch analysis for musical purposes it requires to render an exacting correlation of spectral data to a dynamically comprehensive mathematical pitch model, rationally based upon

Stellenbosch University https://scholar.sun.ac.za

real number relations and their relative deviations from each other in cents and or hertz within tonal contexts.

3.3.2.1 Formulation

Intervals may be expressed by the relations of pitch in hertz values.

Interval $= n : f$
where $f =$ fundamental reference pitch in hertz (hz); and
$n =$ relative sampled pitch in hertz (hz)

These quantities are subject to the law of inverse proportion. The pitch number is inversely proportionate to the sounding body. This is most explicitly rendered visible by the divisions of a string, and the lengths of organ pipes.

Considering that octaves are perfectly equivalent we may convert pitches by doubling or halving their frequency number. The basic principle of doubling and halving relations may be stated mathematically.
$2f = f = {}^1\!/\!_2 f$

To simplify analysis to a single octave we may convert our analysis to a central octave. This is not useful for melodic or harmonic analysis as it inverts many of the relations but it is of use to a gamut analysis that seeks to determine the average or compounded uses of the pitch space within a generalised octave or other interval.

PASS f if $f \geq 256hz$ and $f \leq 512hz$

if $f < 256hz$
then $f \times 2$
until PASS f condition is satisfied

else if $f > 512hz$
then $f \div 2$
until PASS f condition is satisfied

In order to place the relative pitches in ascent above the fundamental frequency it is useful to consider the form of this
transpositional formula.
PASS n if $n > f$ and $n < 2f$

if $n < f$
then $2 \times n$
until $n > f$

Stellenbosch University https://scholar.sun.ac.za

else if $2f < n$
then $n \div 2$
until $2f > n$

Rational tuning relations may be described simply.
if $n = f \times a$
for range, $a = \mathbb{N}$
return $n : f = a : b$
where $b =$ *unity* expressed as $\mathbb{N}$

else if $n = f \times {}^{a}/_{b}$
where $a = \mathbb{N} > b = \mathbb{N}$
return $n : f = a : b$

These may be subject to many more limitations than these simple statements imply. These are outlined further in the software script descriptions in Appendix A.

n-TETs and EDOs may be easily defined.
if $n = f \times \sqrt[a]{b}$
where a and $b = \mathbb{N}$
print $n : f = \sqrt[a]{b} : 1$

The terminological intervals of 12-TET may be correlated, as by the example dictionary entry below.
if $\sqrt[a]{b} = \sqrt[2]{2}$
return $n =$ 12-TET Tritone
else if $\sqrt[a]{b} = \sqrt[3]{2}$
return $n =$ 12-TET Ditone / Major 3$^{\text{rd}}$
else if $\sqrt[a]{b} = \sqrt[3]{4}$
return $n =$ 12-TET Quadritone / Minor 6$^{\text{th}}$
else if $\sqrt[a]{b} = \sqrt[4]{2}$
return $n =$ 12-TET Semiditone / Minor 3$^{\text{rd}}$
else if $\sqrt[a]{b} = \sqrt[4]{8}$
return $n =$ 12-TET Sequiquintone / Major 6$^{\text{th}}$
else if $\sqrt[a]{b} = \sqrt[6]{2}$
return $n =$ 12-TET Whole Tone / Major 2$^{\text{nd}}$
else if $\sqrt[a]{b} = \sqrt[6]{32}$
return $n =$ 12-TET Quintone / Minor 7$^{\text{th}}$
else if $\sqrt[a]{b} = \sqrt[12]{2}$
return $n =$ 12-TET Semitone / Minor 2$^{\text{nd}}$
else if $\sqrt[a]{b} = \sqrt[12]{2048}$
return $n =$ 12-TET Leading Tone / Major 7$^{\text{th}}$

Stellenbosch University https://scholar.sun.ac.za

3.4 Achieving these aims and objectives

The application of these and other formulas forms a simple enough foundation upon which complex rational notions of consonance and musical pitch relations may be established. The complexity arising from such dynamic relational pitch models has been used to develop a methodology that approaches the task of presenting a mathematical summation of musical information in a form useful to musicologists and acoustic researchers.

Mathematically rigourous musical pitch structure, of cents and rational relations, may be mapped to exact pitch location across the fixed frequency spectrum. Information regarding pitch sampling in terms of intonation and temperament, means, averages, regularities and deviations, requires pitches to be tallied over time and regarded in terms of close exaction.

This involves the development of comprehensive dictionary definitions defining the possible mathematical relations of pitch within an octave. These definitions, in terms of logarithmic cents and relative integer and enharmonic ratios, are capable of modelling in an intervallic dictionary. This may serve to resolve a traditional frequency analysis into musically relevant pitch information, to variable resolutions, based upon the hierarchies naturally presented by the various numbered limits and EDO divisions.

Definitions regarding melodic intervals and successions, harmonic compounds of pitch and non-melodic adjustments, deviations and successions, form a set of dynamic dictionaries describing the gamut of tonal octave relations in terminology.

These dynamic context dictionaries, defining occurrences of tonal phenomena such as scalar patterning, intervals, tetrachords, chords, and modulations, are shown in Appendix A. These dictionaries aim to furnish an explicitly defined library capable of allowing for the automated retrieval of musically-meaningful practical information regarding the many contexts of pitch relations.

Stellenbosch University https://scholar.sun.ac.za

Chapter 4

Algorithm and Software Review

Melody has been defined as an auditory object that emerges from a series of transformations along the six dimensions: pitch, tempo, timbre, loudness, spatial location, and reverberant environment [1]

– *Gómez et al. (2003:1)*

4.1 Python

In order to facilitate the translation of this research into a functional software design, this research will make use of Python, an elegant and extensible programming language[2], in order to test and prototype the methodology proposed in this research[3].

4.1.1 The Scipy Stack

This uses the Scipy stack of libraries[4], together with other Python libraries, to extract data arrays from audio files and compute various functions from these arrays.

NumPy is a modul used to deal with arrayed data, and is the Python successor to the Numeric array object [5] and has a rich heritage of open-source community-driven development.

[1] *See Kim et al. (2000)*

[2] `http://www.python.org`

[3] See Glover *et al.* (2011) for more detail regarding the suitability of Python for audio signal processing.

[4] A set of high-level scientific computing extensions for Python. `http://www.scipy.org`

[5] For a complete history of their development and differences, see Oliphant (2006:13-16)

Stellenbosch University https://scholar.sun.ac.za

The Matplotlib library allow for the simple plotting of data from within the Python environment, and also allows for the easy incorporation of Python code into this LaTeXdocument.

Enthought Canopy, providing an interactive environment for Python development, has been the Python Environment of choice for this research. [6]

4.1.2 Bregman Toolkit

The modules contained in the Bregman Toolkit (BregmanTK)[7] constitute a thorough set of tools for test signal synthesis, signal analysis and visualisation.

The tuning module contains well written methods for the creation of data arrays representing Pythagorean tuning system relations, as well as those of Equal Temperaments and Harmonic Intonation[8] and also a simple method for mapping these relations to frequencies in Hertz. The functions offered by these scripts have been further elaborated to create comprehensive extended dictionaries for precision analysis and tabulation of audio data regarding pitch.

The BregmanTK offers a variety of plotting functions, an example is shown in figure 4.1.

4.1.3 LibROSA

The audio extraction technique provided by LibRosa offer a simple protocol to retrieve array information regarding pitch and magnitude over the sampling time of an audio file. It also provides a useful method for limiting the minimum and maximum pitch ranges of output data arrays. These arrays form the basis of the preliminary data array method used in this research to display pitch and magnitude over a time index. Simple unpacking and cross referencing of these arrays provides pitch and magnitude information useful to a mathematical musical analysis, lists of tentative dominant frequencies, and successions of peak frequencies and simultaneous frequencies within bounded magnitudes. These lists are refined here to single octave gamuts which are then processed by correlation with their mathematical and terminological interval dictionary equivalents.

---

[6] `https://www.enthought.com/products/canopy/`

[7] A Python toolkit for audio-visual research and development created by the Bregman Audio Labs at Dartmouth.
`http://bregman.dartmouth.edu/bregman/`

[8] This last, called Just Intonation in the BregmanTK, is implemented according to a Harmonic intonation based upon spectral overtone theory. Though this partially describes Just intonation practices, it must be considered as a distinctly biased contraction of the rational number relations of historic Just intonations.

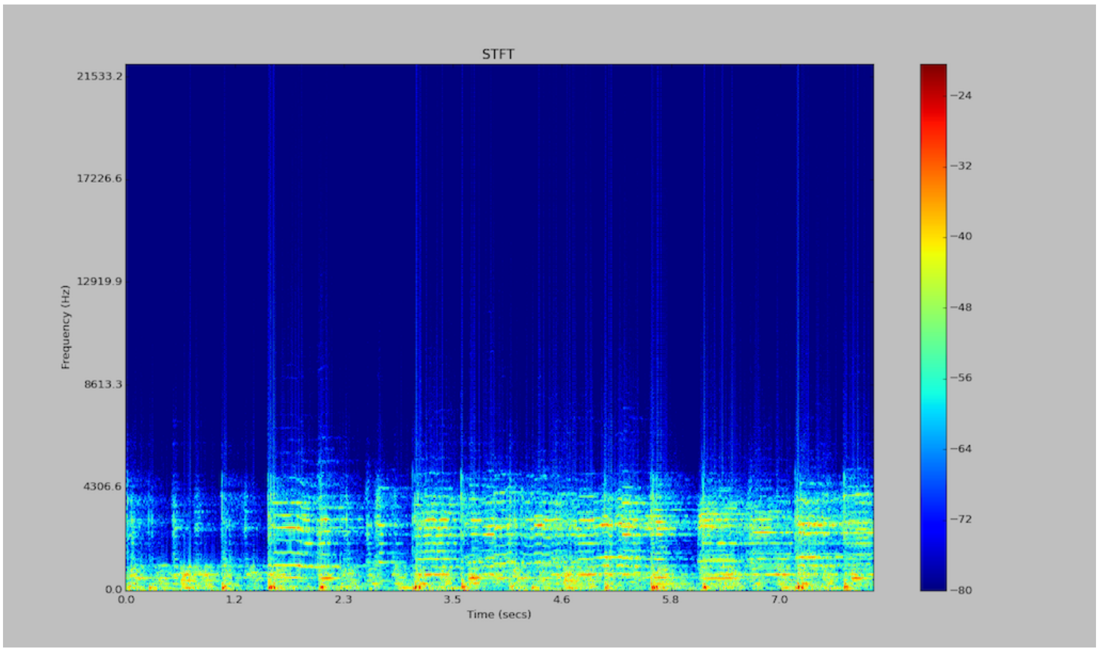

Figure 4.1: Example of Bregman STFT plot

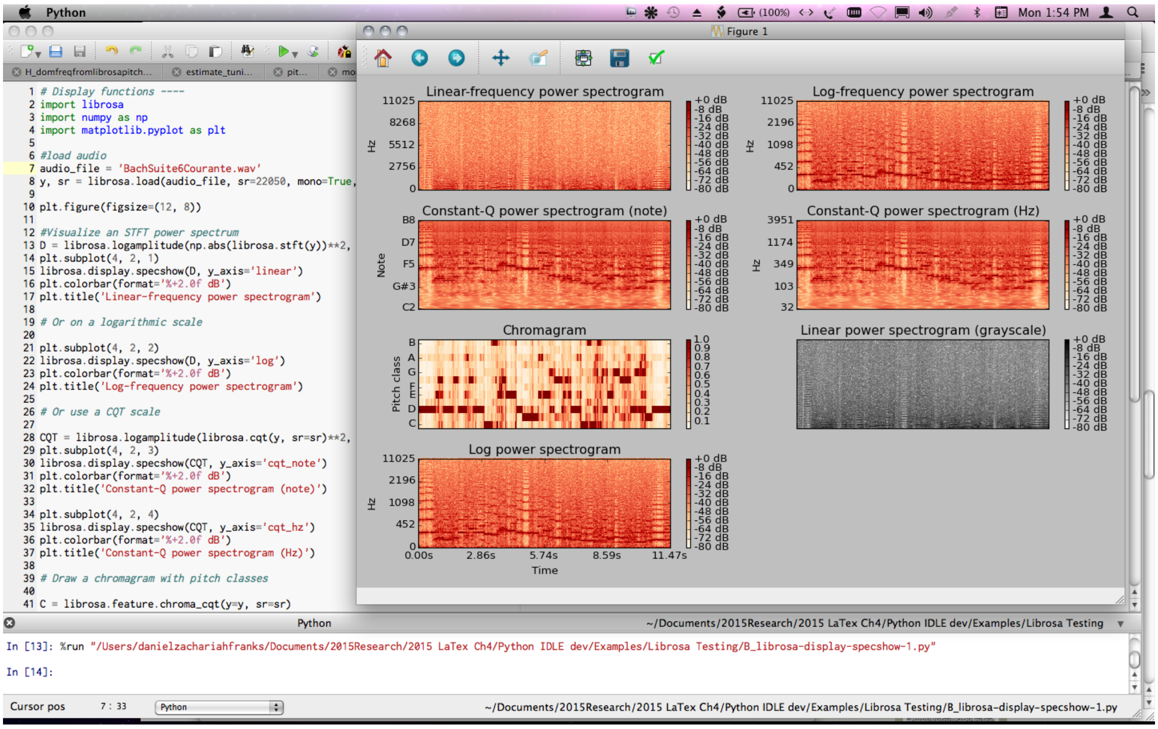

Figure 4.2: Screenshot showing LibRosa Subplotting running in Enthought Canopy's Python environment.

Stellenbosch University https://scholar.sun.ac.za

4.2 Algorithms

Various algorithms are described in this section in relation to their uses for the analysis of musical data.

4.2.1 Melodic Representation

There are many different ways of describing and quantifying melodic pitch for software representation[9].

These may be broadly classified by distinctions between *time-domain* and *frequency-domain* extraction methods. Time-domain algorithms look for periodicity in the time domain of a signal sound. Frequency-domain algorithms look for periodicity in the frequency domain of a signal sound.

From these domains we may draw other distinctions, such as those of *spectral place* and *spectral interval* for frequency domain information, which may be useful in determining the applications of various algorithms for fundamental frequency estimation of melodic pitch succession and harmonic pitch coincidence (Gómez *et al.*, 2003:5-12).

4.2.2 Zero-Crossing Rate

Zero-crossing rate (ZCR) "consists in counting the number of times the signal crosses the 0-level reference in order to estimate the signal period" (Gómez *et al.*, 2003:5). While inexpensive, ZCR does not often yield very accurate data when dealing with "noisy signals or harmonic signals where the partials are stronger than the fundamental" (Gómez *et al.*, 2003:5). ZCR related algorithms are of greater use to timbral research, and are not as well suited for fundamental frequency estimation.

4.2.3 Auto Correlation Function

Auto-correlation function (ACF) based algorithms have proven to be better capable of estimating fundamental frequency over a length of time, and useful to mapping melodic ranges and fundamental frequencies, while also allowing for computation in the frequency-domain using short-time Fourier transform (STFT). (Gómez *et al.*, 2003:6)

[9] "Sometimes melody is considered as a pitch sequence (what we call "*melody as an entity*" trying to answer the question "*Which is the melody of this audio excerpt?*") [...] Melody can also be defined as a set of attributes (what we call "*melody as a set of attributes*" trying to answer the question "*which are the melodic features of this audio excerpt?*") that characterize the melodic properties of sound. Several attributes to be considered are for example key, melodic profile, melodic density (the degree of melodic activity), and interval distribution or tessitura (pitch range). Sometimes melody is also associated with the concept of unity: *an arrangement of single tones into a meaningful sequence.* This definition is close to the concept of phrase." (Gómez *et al.*, 2003:2)

Stellenbosch University https://scholar.sun.ac.za

"The ACF of a sequence $x(n)$ of length K is defined as:

$$r(n) = \frac{1}{K} \sum_{k=0}^{K-n-1} x(k)xk + n$$

Where the maximum of this function corresponds to the fundamental frequency for periodic signals." (Gómez *et al.*, 2003:6)

McLeod & Wyvill (2005) define two types of ACF used to analyse a discrete signal x_t:

$$\text{ACF Type I: } r_t(\tau) = \sum_{j=t}^{t+W-1} x_j x_{j+\tau}$$

Where $r_t(\tau)$ is the ACF of lag τ calculated stating at time index t, where W is the initial window size, *i.e.* number of terms in the summation.

$$\text{ACF Type II: } r'_t(\tau) = \sum_{j=t}^{t+W-1-\tau} x_j x_{j+\tau}$$

In this second type the window size decreases with increasing τ, having a tapering effect and a corresponding smaller number of non-zero terms used in the calculation at greater τ. ACF type I and II are the same for a zero padded set (*i.e.* $x_k = 0, k > t + W - 1$) (McLeod & Wyvill, 2005:2).

When using ACF type II, it is common to divide $r'_t(\tau)$ by the number of terms in order to counteract the tapering effect. "However this can introduce artifacts, such as sudden jumps when large changes in the waveform pass out the edge of the window." (McLeod & Wyvill, 2005:2)

These ACFs are related to the following two discrete signal Square Difference Functions (SDFs).

$$\text{SDF Type I: } d_t(\tau) = \sum_{j=t}^{t+W-1} (x_j - x_{j+\tau})^2$$

$$\text{SDF Type II: } d'_t(\tau) = \sum_{j=t}^{t+W-\tau-1} (x_j - x_{j+\tau})^2$$

However, the minima presented by the SDFs when τ is a multiple of the period, and the corresponding maxima presented by the ACFs, do not always coincide.

Expanding the second type of SDF, we find that it contains an ACF:

$$d'_t(\tau) = \sum_{j=t}^{t+W-\tau-1} (x_j^2 - x_{j+\tau}^2 - 2x_j x_{j+\tau})$$

Defining:

Stellenbosch University https://scholar.sun.ac.za

$$m_t'(\tau) = \sum_{j=t}^{t+W-\tau-1} (x_j^2 - x_{j+\tau}^2)$$

Results in:

$$d_t'(\tau) = m_t'(\tau) = -2r_t'(\tau)$$

ACF analysis may suffer from "twice-too-low" octave errors "since integer multiples of the fundamental frequency n_0 also have positive weights at the harmonics frequencies." (Gómez *et al.*, 2003:6) While this is may present fewer problems for generalised pitch usage analysis, translated to a single octave gamut range, this does pose as a relevant objection to ACF methods by themselves being used to compute melodic and harmonic pitch positions.

> To calculate the SDF by summation takes $O(Ww)$ time, where w is the desired number of ACF coefficients. By splitting $d_t'(\tau)$ into the two components $m_t'(\tau)$ and $r_t'(\tau)$, we can calculate these terms more efficiently. The ACF can be calculated in approximately $O((W + w)\log(W+w))$ time by use of the Fast Fourier Transform. (McLeod & Wyvill, 2005:3)

4.2.4 Fourier Transforms

Jean Baptiste Joseph Fourier contributed to the theory of harmonic analysis his discovery that a periodic wave may be decomposed into a sum of sine and cosine waveforms representing the integer multiples of the fundamental frequency of the periodic wave, each with its own integral amplitude. (Benson, 2007:37)

Fourier transforms are well documented methods for translating a signal into it's frequency-domain constituents. A Fourier transform represents frequency band components as consisting of varying combinations of sine waves. Various implementations of Fourier transforms have resulted in very efficient methods of calculating these functions. Such fast methods are called Fast Fourier Transforms (FFT). These FFTs are used at great lengths in computing musical pitch analysis.

Additionally there are other forms of related Fourier transforms. Short-time Fourier Transforms (STFT), Discrete Fourier Transforms (DFT), and variations on these, the Inverse STFT (ISTFT), Inverse FFTs (IFFT), useful for inverting STFT and FFT frequency-domain back to time-domain, and Real FFTs (RFFT), useful for returning the real part of an FFT.

FFT analysis is an integral part of many Python audio libraries such as LibROSA and the Bregman Audio Toolkit.

For detailed information relating to the mathematics underlying the machine calculation of Fourier Transforms see Cooley & Tukey (1965).

Stellenbosch University https://scholar.sun.ac.za

4.2.5 Cepstrum Analysis

Cepstrum analysis[10] was one of the first algorithms realised by electronic computing systems. Cepstrum analysis works by taking the inverse Discrete Fourier Transform (DFT) of the logarithm of the short-time magnitude spectrum of the signal. While similar to the Autocorrelation systems, Cepstrum differs in the use of the logarithm of the magnitude being used in place of its second power. Cepstrum methods suffer from the same problems as those of ACFs, and may be best considered as a *spectral place* type algorithm (Klapuri, 2000:1) (Gómez *et al.*, 2003:7).

4.2.6 Harmonic Matching and Band-wise Processing

Harmonic matching methods identify the periodicity of a signal from the spectral peaks of its magnitude spectrum, and compares these to predicted harmonics for each possible candidate frequency. These are then differentiated by a fitness measure, introduced in various ways in order to reduce harmonic mismatches Gómez *et al.* (2003:8-9).

Melodic extraction using harmonic matching methods has shown promising results Rao *&* Rao (2008) using a two-way mismatch method described in Maher *&* Beauchamp (1994).

A similar methodology is used in this research to match frequencies to reference pitches generated by mapping various generalised interval relations to the specific dominant frequency determined in the audio sample. These are then correlated to show intersections of audio data to reference data.

4.2.7 Audio Oracle and Machine Learning

Investigations into unsupervised learning, have resulted in the development of structures for indexing aspects of audio data for relational analysis.

The Audio Oracle(AO) algorithm, described by Dubnov *et al.* (2007, 2011) and based upon the Factor Oracle described by Allauzen *et al.* (1999), offers methods for indexing sub-clips of variable lengths. The Audio Oracle's methods of analysis and audio generation have been designed for the purposes of textural synthesis.

---

[10] Cepstrum analysis, first introduced in Bogert *et al.* (1963), deals with a unique permutation of frequency and time-domains. "This new ÒspectralÓ representation domain was not the frequency domain, nor was it really the time domain. So, looking to forestall confusion while emphasizing connections to familiar concepts, Bogert *et al.* (1963) chose to refer to it as the quefrency domain, and they termed the spectrum of the log of the spectrum of a time waveform the cepstrum. While most of the terms in the glossary at the end of the original paper [including such terms as *rahmonics* and *liftering*]have faded into the background, the term cepstrum has survived and become part of the digital signal processing lexicon."(Oppenheim *&* Schafer, 2004:1), for more information see Childers *et al.* (1977),

Stellenbosch University https://scholar.sun.ac.za

The Audio Oracle uses a symbolic mathematical approach to learning sequences and updating transitional logic in relation to such learned sequences.

"a factor oracle [...] is a finite state automaton constructed in linear time and space in an incremental fashion." (Dubnov *et al.*, 2007:2)

"Audio Oracle accepts [an audio] stream as input, transforms it into a sequence of feature vectors and submits these vectors to AO analysis. AO outputs an automaton that contains pointers to different locations in the audio data that satisfy certain similarity criteria, as found by the algorithm." (Dubnov *et al.*, 2007:2).

The methods of Audio Oracle are currently used for purposes of analysis spectral phenomena. These methods may be capable of informing the further spectral analysis of pitch timbre that this research proposes[11], allowing for the discernment of individual voices in polyphonic samples. The iterative learning approach used by the Audio Oracle indicates possibilities for software identification, and learning, of pitch identities, and the differentiation of related phenomena such as combination tones and overtone phenomena.

A Python implementation of the Audio Oracle algorithm is available as PyOracle[12].

The potentials of machine learning have entered musical analysis from the realm of artificial intelligence. Such methods offer scope for complex reiterative analysis and the generation of classifications from such analysis[13].

Machine learning methods have been investigated by musicologists for their ability to model the way in which sound is musically ordered by perception.Lyon (2010:2) describes a "trainable classifier or decision module" for processing extracted musical features. In this stage of analysis, mathematically informed classifications are decided upon, and remembered by the structure to inform future analysis. Such an extension to the dictionary definition and lookup methods described in this study[14] would make for the potentially useful automated learning of mathematical pitch relations and terminological classifications. The blueprint for these decisions and classifications may be drawn form the general principles of n-limit tunings and n-TET orders together with definitions of the various contexts[15] of pitch.

A module for Machine Learning in Python (MLPY)[16] is available based upon the work by Albanese *et al.* (2012).

[11] See chapter 6.

[12] `https://bitbucket.org/pucktronix/pyoracle/downloads`

[13] For further detail regarding these methods see Alpaydin (2014); Carbonell *et al.* (1983).

[14] See chapter 5

[15] Melodic successions, harmonic simultaneities, sympathetic accords contributing to combination-tones, and related tonal contexts influencing and acting upon pitch relations, such as timbral/textural overtone quality and amplitude.

[16] `http://mlpy.sourceforge.net/`

Stellenbosch University https://scholar.sun.ac.za

4.2.8 Super-Resolution

Heightened precision may be achieved by the use of various information enhancing algorithms, though this must be carefully balanced against the possible introduction of inaccurate and irreconcilable data artifacts[17].

In regards to situations requiring the use of super-resolution[18], such as in cases of audio degradation, methods of signal enhancement described by Keegan *et al.* (2011) and Medan *et al.* (1991) may be found useful. Further, situations such as noisy ambiances and other audio artifacts may also benefit from such enhancement.

4.2.9 PHD and MUSIC

The general procedure of dominant frequency extraction may be relatively simply defined as finding the density peak in the power spectrum[19] distribution of the signal to return frequency, then reading amplitude and phase from the signal's Fourier transform. In samples having a small number of fairly distant frequencies and low levels of noise it is indeed so simple. Most practical cases require discussion of a number of further issues, such as the use of predetermined limitations of maximum and minimum frequency and magnitudes, and other methods of classifying accidental or undesirable pitch artifacts. (Telgarsky, 2013:6).

A discussion of various methods and uses for dominant frequency analysis of time series data in Telgarsky (2013:6-9) outlines some specialised algorithms and toolkits available for these purposes, such as Pisarenko Harmonic Decomposition (PHD) and it's generalisation in Multiple Signal Classification (MUSIC) (Telgarsky, 2013:5).

Pisarenko Harmonic Decomposition (PHD) determines amplitudes and frequencies for dominant frequencies, up to a limit specified in advance. MUSIC analyses complex-valued time series as the sum of p complex exponentials and a complex Gaussian white noise and returns p largest peaks (Telgarsky, 2013:5). This approach is similar to the approach used by the MPM algorithm described below in section 4.3.2.1. These may prove useful to further elaborations of the method of pitch analysis proposed in this paper.

[17] For a detailed explanation of audio signal coding see Spanias *et al.* (2006).

[18] "Super-resolution (SR) is the problem of creating a high-resolution (HR) output signal from a low-resolution (LR) input." (Keegan *et al.*, 2011:81)

[19] *Energy spectral density* "represents the distribution of sequence energy as a function of frequency" (Stoica & Moses, 2005:4). "[*Power Spectral Density*] 'measures' the power at a frequency in the signalÕs [autocovariance sequence (ACS)]" (Stoica & Moses, 2005:9). ACS defines the covariance function of the finite-energy sequence $y(t)$ Stoica & Moses (2005:4-5).

Stellenbosch University https://scholar.sun.ac.za

4.2.10 Wavelet Analysis

> [T]he wavelet transform (WT) is a multiresolution, multi-scale analysis that has been shown to be very well suited for music processing because of its similarity to how the human ear processes sound. In contrast to the STFT, which uses a single analysis window, WT uses short windows at high frequencies and long windows for low frequencies. This is the spirit of the constant $Q(\frac{\Delta f}{f_c})$ frequency analysis. Gómez *et al.* (2003:9)

Wavelet analysis as a method of audio encoding may offer some advantages over traditional Fourier transforms for the purposes of temporal and spectral encoding.

Wavelet analysis consists of a diffusely represented two dimensional spectrum (*e.g.frequency* and *time*) transform generated from a one-dimensional time series or frequency spectrum.

For the quantitative purposes considered in this research, the importance of obtaining statistically significant results must be taken into consideration. Significant relations may be disguised by the two-dimensional result of the wavelet transform. Often this is further complicated by the application of arbitrary normalisation with the result that "the wavelet transform has been regarded by many as an interesting diversion that produces colorful pictures, yet purely qualitative results". (Torrence *&* Compo, 1998:1)

However, it's advocates argue that Windowed Fourier Transforms (WFT) "represent[...] an inaccurate and inefficient method of time-frequency localization, as it imposes a scale or "response interval" T into the analysis. The inaccuracy arises from the aliasing of high and low frequency components that do not fall within the frequency range of the window." (Torrence *&* Compo, 1998:3) This is an interesting flaw to consider, as such assumptions regarding scale underlie many approaches to analysis and instances of sampling error.

"For analyses where a predetermined scaling may not be appropriate because of a wide range of dominant frequencies, a method of time-frequency localization that is scale independent, such as wavelet analysis, should be employed." (Torrence *&* Compo, 1998:3)

In order to analyse time series containing multiple instances of non-stationary power at different frequencies, the wavelet transform may be usefully applicable. For a time series x_n, with equal time spacing δt and $n = 0...N-1$, there is a *wavelet function*, $\Psi_0(\eta)$, where η is a non-dimensional "time" parameter. An "admissible" wavelet function has a mean of zero and is localised in both time and frequency[20]. (Torrence *&* Compo, 1998:3)

[20] An example is the Morlet Wavelet, consisting of a Gaussian modulated plane wave:

" $\Psi_0(\eta) = \pi^{-1/4} e^{i\omega_0 \eta} e^{-\eta^2/2}$
where ω is the non-dimensional frequency [...]"
(Torrence *&* Compo, 1998:3)

Stellenbosch University https://scholar.sun.ac.za

Wavelet functions may refer to either orthogonal or nonorthogonal wavelets. *Wavelet basis* refers only to orthogonal sets of wavelet functions. An orthogonal basis implies a *discrete wavelet transform*, and a nonorthogonal function can refer to either discrete or the *continuous wavelet transform*. (Torrence & Compo, 1998:3-4)

A discrete sequence x_n is defined as "the convolutions of x_n, giving a scaled and translated version of
$\Psi_0(\eta)$: $W_n(s) = \sum n' = 0^{N-1} x'_n \Psi * [\frac{(n'-n)\delta t}{s}]$
where $*$ is the complex conjugate.

"[V]arying the *wavelet scale s* and translating along the *localized time index n*, one can [...] show [...] both the amplitude of any feature versus the scale and how this amplitude varies with time." (Torrence & Compo, 1998:4)

Dropping the subscript 0 on Ψ indicates its normalisation.

Possible applications for using wavelet analysis may be further scrutinised for potential in furthering the method of musical pitch data extraction this study seeks to define. Their use in provided for in SciPy, while PyWavelets[21] provide open-source wavelet transform extensions to these.

4.3 Software

4.3.1 Dynamic Tonality

The model of *Dynamic Tonality*, as defined by Milne *et al.* (2007); Sethares *et al.* (2009), proposes "a novel way of organizing the relationship between a family of tunings and a set of related spectra"(Sethares *et al.*, 2009:3), treating of pitch as defined in terms of metrics and tensor vectors. The propositions involved in this modelling of tonality supposes that a numerical, mathematical model of pitch relations and functions, with a corresponding computational structure, may provide a useful step towards a more *parsimonious*, or inflexible, model for defining tonal pitch relations, arguing further that:

> An advantage of a good-fitting parsimonious model over a good-fitting non-parsimonious model is that only the former generalizes beyond the specific sample of data to which they are fitted. (Milne, 2013:17)

The model of Dynamic Tonality has been used by Sethares and Milne to develop of a series of microtonal synthesizers, 'TransFormSynth', 'the Viking', '2032', and the sequencer 'Hex'[22]. These software tools allow musicians to

[21] `http://www.pybytes.com/pywavelets/`

[22] A stand-alone software application built using Cycling 74's Max/MSP(Max Signal Processing). Max was originally written by Miller Puckette, who went on to create PureData, the open-source alternative to Max.

Stellenbosch University https://scholar.sun.ac.za

experiment, perform and compose using a multitude of temperaments and tuning schemes, enabling complex manipulations of these using a relatively small number of parameters. Translation between tuning schemes, and the operations involved, are made visible, and variable, by the use of simple parameters and the mapping of pitches to a two-dimensional isomorphic note layout. This representation of pitch relations is modelled after the Tonnetz[23] model, and is realised in various enharmonic keyboard layouts, making interesting use of tonal-cardinalities in the directional representation of intervals. The resulting symmetrical interval arrangements bear great likeness to the enharmonic keyboard designs of Bosanquet.(Prechtl *et al.*, 2012:1-2)

Further detail regarding these keyboard topologies and their histories are to be found in Milne *et al.* (2008), Hiebert (2014) and Helmholtz (1895:429). It may be noted that these layouts are in many ways adequate for practical usage by fixed-pitch practices, and due to the directional variety of pitch placements they are in some ways more adequate than other button based tuning scheme. However, the parametric nature of their fixed pitch relations puts these layouts at the same disadvantage as any other fixed-pitching instrument, namely that continuous graduations of pitch are only possible by the manipulations of some additional parameter. This is in contrast to the continuum of pitch placements provided by string lengths with moveable stops, or other freely pitched instrument methods.

These layouts proceeded from enharmonic experiments with organ stops and string instruments that preceded the general introduction of equal-temperament, and the common usage of twelve tone keyboard manuals and other instruments having twelve fixed pitch classes[24].

In these developments, mathematical parameters defining relations were developed to allow for changes to the pitch of the sound of the organ pipes, or string length, by the use of a keyboard button arrangement. Additions of parameters and associated mechanisms to the organ keyboard layout were often made in order to achieve some desired manipulation of the sound produced. Many string keyboard instruments, such as the clavichord[25], made allowances for tuning variations in tuning individual enharmonic identities. Desired manipulations ranged from the timbral, such as *diapason*[26] stops being introduced, to intonational shifts, changing the vibration number of the

[23]The Tonnetz is a model which may take a note-based representation, consisting "of a circle of n-dimensional cubes linked by shared vertices", or a scheme of chord relations consisting "of a circle of n-dimensional cubes linked by shared facets" (Tymoczko, 2012:1-3).

[24]See section 2.2

[25]A tangent struck string instrument with a keyboard mechanism. See Brauchli *et al.* (1998); Thwaites *&* Fletcher (1981) for further detail.

[26]This doubling the fundamental note at its octave. A similar mechanism for string instruments involves the use of a *double course* of two strings in place of a single string, these being tuned an octave apart.

Stellenbosch University https://scholar.sun.ac.za

pitch by a comma or some other small interval[27].

These layouts may permit of various convenient permutations of simultaneous harmony, but they do not permit of the many subtle shades of portamento and other grace ornaments that are common, indeed necessary according to many schools, aspects of the musical movements of a single voice. Without very specific arrangements for touch sensitive keyboards[28], there can be no natural vibrato, as of a voice or a string. So too, there can be no true legato, however much a masterful technique may succeed in disguising this inability, pitch movements upon fixed pitch instruments, especially those without powers of sustained and gradual portamento, amount to staccato leaping from fixed point to fixed point.

An interesting timbral aspect of Dynamic Tonality lies in the possibilities for tempering partial tones, using a mapping of the underlying scale to 'temper out' inharmonic partial tones. This introduces many possibilities of timbre that are otherwise very difficult to attain and reduces the extent of certain undesirable inharmonic attributes of temperament. (Prechtl *et al.*, 2012:2) Such an approach, while reducing the drawbacks to using tempered button layouts, does not reduce the difficulty regarding the exact placement of their embedded pitches. This last aspect, regarding the exact mathematical relations of pitches, and the matter of their realisation, is an important issue deserving of further consideration.

Tuning schemes are realised in Dynamic Tonality by the modelling of *r-dimensional tuning* schemes as systems of intervals generated by combinations of r number of independent (incomposite) intervals, or *generators*, where it is understood that *linear independence* represents a logarithmic scale of inequality. (Prechtl *et al.*, 2012:4) Milne *et al.* (2008)

Such r-dimensional tunings are conventionally represented by the sizes of their generators notated in cents.

> [...] all n-TETs are 1-D tunings and vice versa. (Prechtl *et al.*, 2012:5)

12-TET is a 1-dimensional tuning where the only incomposite interval is the equal-tempered semitone of 100 cents. Similarly 24-TET is a 1-dimensional tuning where the only incomposite interval is the equal-tempered quarter-tone of 50 cents. Similarly, 4-TET is a 1-dimensional tuning where the only incomposite interval is the equal-tempered semiditone of 300 cents, and 3-TET

[27] See Hawkins (1853) and Keislar (1988) for further explanation of historic experiments regarding instruments and the realisation of pitch theories.

[28] Just such an arrangement as constituted the historic clavichord's direct tangent-key operation of a single string, allowing for very subtle inflection and intonational discretion, and also allowing access to numerous divisions of a single string by the placement of its numerous keyed tangents.

Stellenbosch University https://scholar.sun.ac.za

is a 1-dimensional tuning where the only incomposite interval is the equal-tempered ditone of 400 cents. [29] 2-dimensional tunings are those such as the Pythagorean, or 3-limit, consisting of all the forms of $2^x \times 3^y$ where x and y are $\mathbb{N}$. Pythagorean / 3-limit tuning has as its generator the perfect fifth of $3/2$, approximately 702 cents, and its octave inversion, the fourth of $3/2$, or approximately 498 cents. From these basic intervals, successive chromatic degrees are constructed after the fashion of the circle of fifths/fourths, yielding a multitude of quantifiable enharmonic differences. 3-dimensional tunings are those such as the Ptolemaic, or 5-limit, consisting of all forms of $2^x \times 3^y \times 5^z$ where x, y and z are $\mathbb{N}$ (Prechtl *et al.*, 2012:5). Ptolemaic/5-limit tuning adds to the Pythagorean pitch set described above, contributing pitch sets obtained by deriving additional enharmonic identities from successive thirds / sixths and their multiplication/addition to the intervals of the 3-limit class. The additional generators contributed by the 5-limit class are principally the intervals of the major and minor thirds, $5/4$ or approximately 386 cents and $6/5$ or approximately 316 cents respectively, and their related octave inversions, the major and minor sixths of $5/3$ or approximately 884 cents and $8/5$ or approximately 814 cents respectively. Similar relations of numbered limits and n-TET schemes, and the corresponding parameters of dimensionality and generators, are utilised by the methods of Dynamic Tonality, not just to achieve various pitch mappings upon the isomorphic keyboard layouts, but also in an attempt to correlate the dimensionality of the keyboard layout to the dimensions of various tuning and temperament schemes.

In principal, many of these layouts function much like the keys in an accordion button layout, in that fifths ascending and fourths descending are arranged along some principal cardinality in succession. This principal cardinality is enlarged by further rows of keys arranged parallel to the first but offset slightly to permit diagonal movements, and two diagonal cardinalities in place of a single perpendicular cardinality. The function of these additional rows serves most often to translate the pitches of the principal cardinality by thirds major and minor, resulting in the diagonal cardinalities describing augmented triads along one axis and diminished quartads along the other.

These layouts serve to greatly enlarge possibilities for enharmonic pitch positions upon fixed pitch instruments, however their general acceptance has been hampered by the great theoretical complications underlying any practical attempts to write for such instrumentation, and indeed similar difficulties encountered in performing upon such keyboard layouts.

[29] These last two mentioned describe the 'diminished' quartad and 'augmented' triad subsets of the pitches available in 12-TET.

Stellenbosch University https://scholar.sun.ac.za

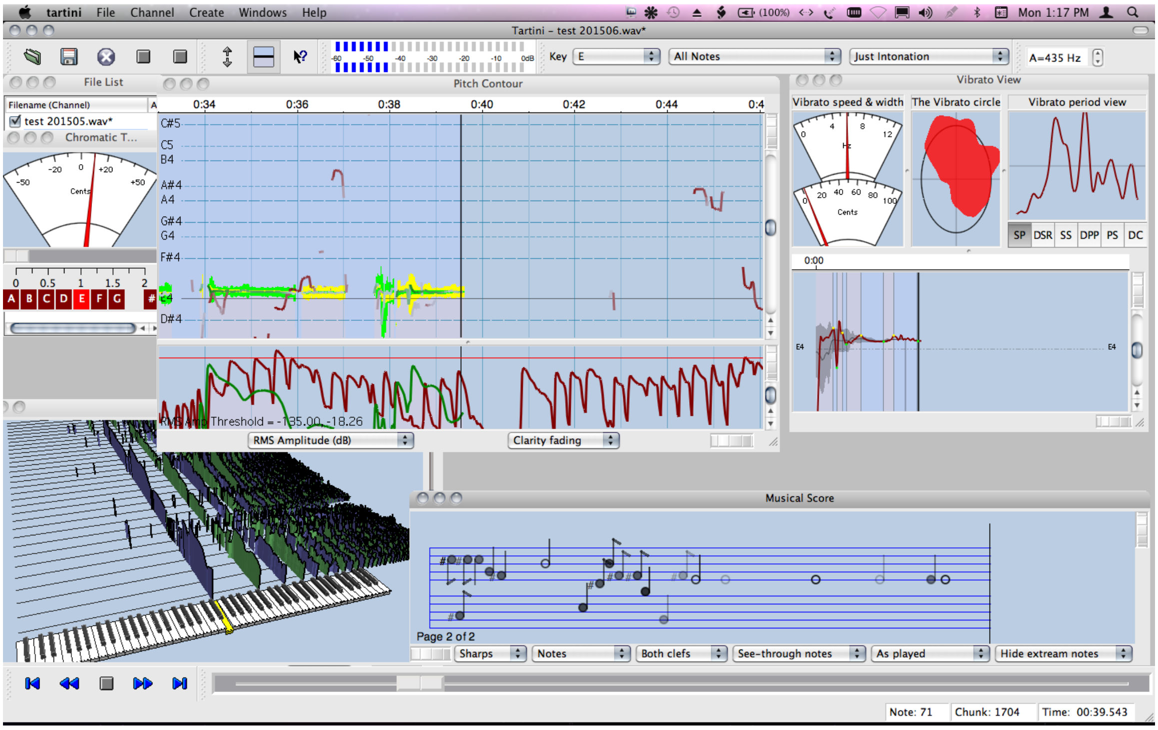

Figure 4.3: Tartini Screenshot

4.3.2 Tartini

Tartini[30], with its user friendly real-time graphical display has notably been found to be of use in the instruction of expressive, and florid, musical practices in arabic singing (Al-Ghawanmeh *et al.*, 2014:3). The graphical user interface provides a neat melodic graph indicating the procession of sounds and their relative height, with a special vibrato window and pitch height graph depicting the real-time analysis of each continuous sound.

Currently, Tartini allows only for monophonic pitch-tracking, making it useful for tracking a single voice but less useful for harmonic values, however its methods of fundamental pitch tracking may be found useful for informing further refinement to the fundamental pitch tracking methods proposed in this research.

Tartini makes use of the McLeod Pitch Method (MPM) to track changes in fundamental pitch.

4.3.2.1 McLeod Pitch Method

> MPM runs in real time with a standard 44.1 kHz sampling rate. It operates without using low-pass filtering so it can work on sound

[30] `http://miracle.otago.ac.nz/tartini/`

Stellenbosch University https://scholar.sun.ac.za

> with high harmonic frequencies such as a violin and it can display pitch changes of one cent reliably. MPM works well without any post processing to correct the pitch. Post processing is a common requirement in other pitch detectors. (McLeod & Wyvill, 2005:1)

MPM makes use of a fast calculation of the Normalised Square Difference Function (NSDF), which is achieved by calculating Square Difference Functions (SDFs) from Auto Correlation Functions (ACFs) rendered by carefully windowed FFT processing. This results in the determination of the dominant waveform of the signal at any given time.

The MPM algorithm may be summarised by the following five step process. (McLeod & Wyvill, 2003:2)

1. Select a sampling window from the incoming data. For continuous display, these windows may overlap in time.
2. Apply the Gaussian function to the window.
3. Perform the FFT. Identify principal frequencies.
4. Identify the fundamental as a sub-multiple of the frequency of greatest amplitude.
5. Recognise the fundamental as a note of the musical scale.

The MPM algorithm is currently tailored towards the analysis of monophonic sound. As the results of this research, a set of mathematically thorough software descriptions of pitch relations for computation, are equally applicable to both harmonic and melodic description, methods should be found to incorporate such of the MPM methods as may be found useful without excluding potentials for harmonic analysis.

4.3.2.2 Windowing

> Existing pitch algorithms that use the Fourier Domain suffer from spectral leakage. This is because the finite window chosen in the data does not always contain a whole number of periods of the signal. The common solution to this is to reduce the leakage by using a windowing function, smoothing the data at the window edges. This requires a larger window size for the same frequency resolution. A similar problem happens in some time domain methods, such as the autocorrelation, where a window containing a fractional number of periods, produces maxima at varying locations depending on the phase of the input. MPM however, introduces a method of normalisation which is less affected by edge problems. Keeping track of terms on each side of the correlation separately. (McLeod & Wyvill, 2005:1)

Stellenbosch University https://scholar.sun.ac.za

Windowing of signal data is crucial to the determination between noise and tone and the presentation of musical tone as periodic phenomena. Careful windowing helps to prevent unwanted artifacts resulting from window overlap, and increases the resolution of valuable phenomena which may be obscured by less painstaking methods of windowing signal data.

In the MPM, having calculated the SDF at time t, it remains to determine which coefficient correctly corresponds to the phenomena of pitch. This is achieved by the NSDF making distinctions between overall and local minima, where only some local minima reflect the changing state of harmonic and melodic pitch. Overall minima reflect less accurate averages of pitch data and many unnecessary data peaks.

SDF type II is used by the MPM in order to define a *symmetry property*, determining local minima by using the same number of evenly spaced samples from both sides of time t, these samples are symmetrical in their respective distances from t. This method "maximises cancellations of frequency deviations from opposite sides of time t, creating a frequency averaging effect." (McLeod *&* Wyvill, 2005:2)

MPM algorithms define key maxima from correlation coefficents at integer τ. From these key maxima, thresholds are defined "equal to the value of the highest maximum, multiplied by a constant." (McLeod *&* Wyvill, 2005:3)

This threshold is of great interest to the methodology offered in this research as it makes some valuable method for defining pitch ranges and acceptable thresholds for pitch deviations and errors.

> Pitch is a subjective quantity and impossible to get correct all the time. In special cases, the pitch of a given note will be judged differently by different expert listeners. We can endeavour to get the pitch agreed by the user/musician as most often as possible. The value of [the threshold constant] can be adjusted to achieve this, usually in the range 0.8 to 1.0. (McLeod *&* Wyvill, 2005:3)

Further aspects of MPM analysis are useful for considerations they offer to pitch analysis and relations of pitches. Defining pitch period in terms of sample rate, MPM utilises methods of logarithmic reduction of pitch comparison in a process analogous to the analysis by pitch numbers and cents used by Ellis (1863*a*:95-6)[31].

> The pitch period is equal to the delay, [...] at the chosen key maximum. The corresponding frequency is obtained by dividing the sample rate by the pitch period (in samples). We turn this into a note on the even tempered scale [32] [corresponding] to notes on the

[31]See section 3.3.1.

[32]Using: note $= \frac{\log 10(f/27.5)}{\log 10(\sqrt[12]{2})}$

Stellenbosch University https://scholar.sun.ac.za

> midi scale, and [containing] decimal parts representing fractions of a semitone. (McLeod & Wyvill, 2005:3)

Different windowing of data provide differing interpretations of data. This consideration becomes especially important for analysis involving measuring frequency. The dynamic windowing methods used by the MPM algorithm deserve further investigation for how these methods may help to inform further development of the methods described in this research.

4.3.3 PsySound 3

PsySound 3[33], for Matlab[34], has developed methods of treating pitch analysis that allow for saliences of out-of-pitch ranges to be expressed and has also gone some way towards realising the mapping of harmonic content and information regarding articulation (Cabrera *et al.*, 2007:5-8).

PsySound provides psychoacoustic models, tools, and measurements, to researchers, and offers a modular extensibility allowing users to write new functions using Matlab's programming environment (Cabrera *et al.*, 2007:2).

The physical analysis procedure of PsySound allows for the extraction of sound-level meter functions, such as sound pressure levels, modelled as time series, as well as Fourier transforms, cepstrum analysis, Hilbert transforms, and auto-correlation functions. (Cabrera *et al.*, 2007:3)

PsySound also provides for both dynamic and steady state loudness modelling[35] and provide various auditory filters. Specific loudness, the loudness of particular frequencies or bandwidths is made use of to determine the extent of possible masking due to imbalances of loudness and to determine spectral loudness patterns. Specific loudness also helps to inform the measure of spectral brightness, or sharpness, as " a subjective measure of sound on a scale extending from dull to sharp". Loudness fluctuation modelling[36] is used to further inform the approach to loudness modelling (Cabrera *et al.*, 2007:3-4).

While the documentation makes some mention of the more elaborate approaches of Shepard (1982), the methods used by PsySound 3 to account for degrees of roughness and dissonance and model perceived pitch analysis are drawn from the pitch model of Terhardt *et al.* (1982), "based on frequency domain template matching of harmonic series rather than auto-correlation". In accordance with this model, Psysound 3 quantizes pitch results to fit the categories of 12-TET, "sharing saliences of out-of-tune pitches between adjacent pitch categories", though it does not implement Terhardt's suggested pitch

[33] `http://www.psysound.org/`

[34] `http://www.mathworks.com/`

[35] Steady state modelling accounts for spectral effects on loudness, while dynamic models "also account for the effect of auditory temporal integration on loudness" thereby providing a higher detail of time-varying signal analysis (Cabrera *et al.*, 2007:3).

[36] See JND in glossary for some explanation of how loudness affects pitch perception.

Stellenbosch University https://scholar.sun.ac.za

shifting as this last would degrade the exactitude of the results. “Pitch salience patterns are expressed linearly over the pitch height range, and circularly over the chroma range.” (Cabrera *et al.*, 2007:4-5) Using this model Psysound calculates the most likely tonic and projects potential harmonic content and functions to fit that tonic. (Cabrera *et al.*, 2007:5-6)

Articulation information is calculated based on an Average Silence Ratio[37]. The wide range of articulation and the dependencies of subjective contexts render this a difficult area of analysis, and this aspect of PsySound is currently undergoing development informed by the statistical method indicated by Brosbol & Schubert (2006)[38] (Cabrera *et al.*, 2007:6).

Datawise, Psysound 3 follows an object-oriented approach to data storage. Broadly speaking, these object formats fall into three classes, time-series objects, spectrum objects and time-spectrum objects. The time-series object is the simplest, consisting of a single stored value changing over time. A spectral object indicates data that is two-dimensional but does not change over time. A time-spectrum object is likewise a spectrum object whose value changes over time. (Cabrera *et al.*, 2007:6)

The purposes PsySound 3 serves are those of psycho-cognitive research, and its developments, notably those of explicit and robust pitch salience models, provide useful methodology for integrating psycho-cognitive investigation with the psycho-acoustic observations of sound relations.

4.3.4 MIRtoolbox

The MIRtoolbox[39] of functions written in Matlab may also provide useful models for extending this research methodology, not the least for its approach to deriving and describing pitch classes and tonality. In addition to chromagram analysis, the MIRtoolbox contains functions for deriving key strength and describing Self-Organising Maps (SOM), or Kohonen Maps[40], describing “key relations that correspond to music theoretical notions” (Lartillot & Toiviainen, 2007:3).

MIRToolbox also makes use of functions included in several public-domain toolboxes, Auditory Toolbox[41], NetLab[42] and SOM-toolbox

[37] Determined by comparison of relative durations and silences Feng *et al.* (2003)

[38] In this method an Articulation Index (AI), against which note durations may be compared, is calculated by dividing the amount of time during which the notes sound by the total length of the piece Brosbol & Schubert (2006:1365).

[39] `https://www.jyu.fi/hum/laitokset/musiikki/en/research/coe/materials/mirtoolbox`

[40] See Kohonen (1998, 2001).

[41] Slaney (1998)

[42] Nabney (2002)

Stellenbosch University https://scholar.sun.ac.za

4.3.4.1 Auditory Toolbox

The Auditory Toolbox implements six methods of auditory time-frequency representation.

1. The computational model described by Richard F. Lyon[43] offers a relatively simple modelling of the complex functions of the cochlea. The main simplification of this method consists in the separation of the interacting functions of the organs of Corti and and the basilar membrane. This is followed by several stages of filtering, detection, compression and neural representation. (Lyon, 1982:1) "This model can represent sound at either a fine time scale (probabilities of an auditory nerve firing) or at the longer time scales characteristic of the spectrogram or Mel Frequency Cepstrum Coefficient (MFCC) analysis." (Slaney, 1998:3)

2. A model of psychoacoustic filtering based on critical bands proposed by Holdsworth *et al.* (1988) combines a description of a Gammatone filter bank with a model of hair cell dynamics[44].

3. A cochlear model, described by Seneff (1986), combining a critical band filterbank with methods of "detection and automatic gain control".

4. Narrow-band and wide-band spectrogram FFT analysis.

5. A speech recognition system, composed of Mel-frequency cepstral coefficients (MFCC), whose "technique combines an auditory filter-bank with a cosine transform to give a rate representation roughly similar to the auditory system."

6. The linear-predictive analysis often used to model a speech signal by conventional speech-recognition systems is also included. (Slaney, 1998:3)

[43] As Lyon (1982:1) puts it: "[...]*The proposed computational models of human audition*] concern mechanical filtering effects and the mapping of mechanical vibrations into neural representation. [*This computational model of filtering, detection and compression in the Cochlea*] cleanly separates these effects into time-invariant linear filtering based on a simple *cascade /parallel filterbank* network of second-order sections, plus transduction and compression based on half-wave rectification with a nonlinear *coupled automatic gain control* network." This model was first explored by Lyon (1978) in an unpublished paper on a signal processing model of hearing which became the basis for his preliminary formal investigation of the computational modelling of the cochlea Lyon (1982). This was later further refined in a description of an analog electronic cochleaLyon *&* Mead (1988) and his exploration of filter cascades as analogs of the cochleaLyon (1998). More recently, this work has developed into a DSP approach to machine-hearingLyon (2010).

[44] This last aspect is based on the inner-hair cell and auditory nerve complex proposed by Meddis *et al.* (1990); Sumner *et al.* (2002).

Stellenbosch University https://scholar.sun.ac.za

Many of the functions in the Auditory Toolbox for Matlab are usefully modelled in Python by the Python Auditory Modelling Toolbox (PAMbox) library.[45]

4.3.4.2 SOM Toolbox

The SOM Toolbox for Matlab[46] is based upon the data mining applications of Self-Organising Maps (SOM)[47] and their application to facilitating unsupervised learning. (Vesanto *et al.*, 1999:1)

> The Self-Organising Map represents the result of a vector quantization algorithm that places a number of reference or codebook vectors into a high-dimensional input data space to approximate to its data sets *in an ordered fashion*. When local-order relations are defined between the reference vectors, the relative values would lie along an "elastic surface". By means of the self-organizing algorithm, this "surface" becomes defines as a kind of non-linear regression of the reference vectors through the data points. A mapping from a high-dimensional data space $\mathbb{R}^n$ onto, say, a two-dimensional lattice of points is thereby also defined. Such a mapping can effectively be used to *visualize* metric ordering relations of input samples. In practice, the mapping is obtained as an asymptotic state in a learning process. (Kohonen *et al.*, 1995:5)

The SOM Toolbox offers many options for the relatively simple visualisation of high dimensional data[48] upon graphs and matrices.(Vesanto *et al.*, 1999:6) The applications of SOM for describing heavily populated and clustered datasets and their properties has been investigated in Vesanto *&* Alhoniemi (2000) with the result that a two stage procedure was found by them to be highly efficient for the purposes of data surveying and ordering. In the first stage SOM compares various conventional *means* (mathematically mean proportions) for those descriptions that most accurately partition the data for the purposes of the survey, and in the second stage these selected means and their mathematical descriptions are clustered to produce datasets describing the sample for purposes of representing various attributes of the data.[49]

[45] `http://pambox.org`

[46] Available free under GNU General Public License at http://www.cis.hut.fi/projecs/somtoolboxnd fully documented in Vesanto *et al.* (2000).

[47] Also known as Kohonen Maps, after their description by Kohonen (2001).

[48] See Vesanto (1999).

[49] The in-depth applications of SOM to data mining, especially concerning neural networking (learning may usefully be treated as a form of neural networking), are further described in Vesanto (2000).

Stellenbosch University https://scholar.sun.ac.za

(Vesanto & Alhoniemi, 2000:1) The SOM intermediation of the data, as opposed to direct clustering of the data, has been found to be more efficient in terms of computational cost, suggesting it's particular usefulness in conjunction with other algorithms. (Vesanto & Alhoniemi, 2000:14) The similarities of this approach to those suggested in this study warrant further investigation towards the integration of these SOM tools.

A SOM module for Python is available as SOMpy[50] and a comparable Kohonen module for vector quantization is also available for Python[51].

4.3.5 Notation

The accurate representation of information is the key task of any notation. This goal is especially remarkable in regards to written score and intonation data. There are a variety of related modules for conventional western musical notation available for Python[52]. While there is a notable lack of specialised software dealing with the explicit notation of Helmholtz-Ellis (HE) notation, an extension permitting HE notation is available for Lilypond[53]. This may allow for the further development of musical notation output from the mathematical analysis proposed in this research[54].

4.3.5.1 Scala

Scala[55] offers many scale definitions and utilities for plotting and comparing scales and pitch sets. The Scala file format is also supported by a number of score-editors and sequencers and is cross-compatible with the Python modules for notation mentioned in section 4.3.5.

A list of over 4500 scale files is available for Scala, and some of these definitions have been used to fill in the scale dictionaries elaborated by this research. However, the terminology used in this research to represent mathematical or-

---

[50] `http://paraschopra.com/sourcecode/SOM/`

[51] `https://pypi.python.org/pypi/kohonen/1.1.2`

[52] Abjad, available at `http://www.projectabjad.org/`; Frescobaldi, available at `http://www.frescobaldi.org/`; and Mingus, available at `http://code.google.com/p/mingus/`, are all Python wrappers for Lilypond, a free text-based sheet-music editor, available at `http://lilypond.org/`.

[53] Created by Torsten Anders, the HE extension for Lilypond is available at `http://cmr.soc.plymouth.ac.uk/tanders/software.htm`.

[54] A number of other 'microtonal' notation options are mentioned at `http://xenharmonic.wikispaces.com/Software`, however none of these are as distinctly aligned to the HE notation and its mathematically explicit representation of pitch relations. See section 3.1.4

[55] A graphical command line environment for experimentation with musical tuning. Available from `http://www.huygens-fokker.org/scala/`

Stellenbosch University https://scholar.sun.ac.za

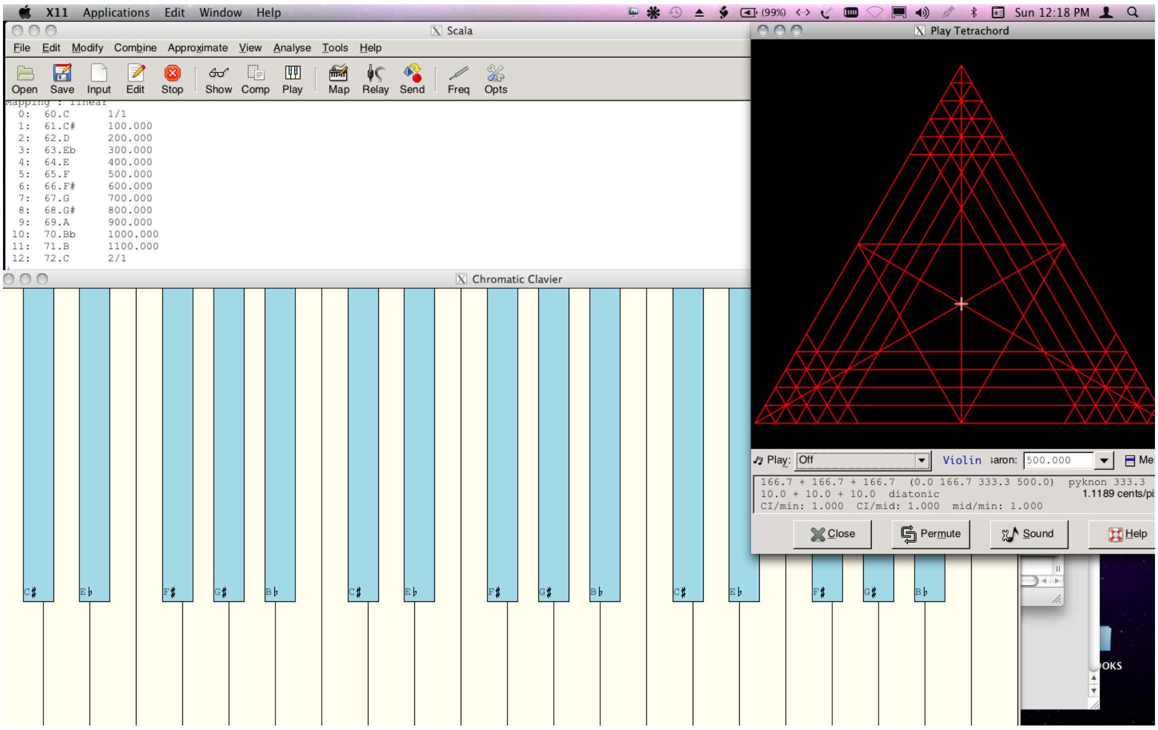

Figure 4.4: Scala Screenshot

ders of pitch sets differs markedly from the terminology provided in these scale files[56].

4.4 Utilisation

The investigations into Python software above have been used to inform the development of the scripts described by this research. The various approaches to frequency analysis exhibited by these algorithms have served to inform the scripted examples and in many cases form integral components of the libraries referenced and used in the development of these scripts. The various software suites investigated above offer approaches to musical analysis that were useful to informing the approach of this research to frequency analysis and providing meaningful musical analysis. The approaches to musical analysis described by these suites above[57] are also useful for the comparisons they offer relative to the approach described by this research. In this last respect, compared to

[56]Notably, this research uses a simplified hierarchy of mean relations to order the rational and irrational tuning systems and a correspondingly simple descriptive structure. See chapters 3 and 5.

[57]Notably Psysound, Tartini and the Audio Oracle Toolkit.

Stellenbosch University https://scholar.sun.ac.za

these commercial suites, this research is seen to offer a greatly enlarged scope for a mathematically exact dictionaries of terminology and reference hierarchy.

In this research Python was used extensively to prototype the scripts utilised to showcase the mathematical descriptions defined in this paper. Enthought Canopy was used as an environment in order to write, iterate, and otherwise troubleshoot the functionality of the scripts developed by this research. The numerous libraries and algorithms described above, especially those forming part of the libRosa libraries and the Bregman Audio Toolkit, formed a key part of the object based scripts described in this research. The harmonic matching techniques and various spectrum analysis methods used in these libraries informed the approach to discerning frequencies from audio input and defining test libraries of relations and terminological dictionaries. The wide variety of python-compatible audio-toolkits and notation suites, while not all utilised in the scripts developed in this research, offer a view to the future development of these scripts.

Stellenbosch University https://scholar.sun.ac.za

Chapter 5

Proposed Method of Data Processing and Music Information Retrieval

This research presents a complex method for describing generalised tonal usage, contributing useful descriptions for the software analysis of melodic successions, harmonic densities, and distinguishable non-melodic/non-harmonic pitch variations. These utilise the correlation of extensible reference dictionary definitions to the results produced by a pitch analysis of audio data.

5.1 Proposed Methodology

THIS methodology proposes a software analysis of audio data with the aim of extracting the relations of their pitches over time and to further relate these to a series of reference dictionaries defined by mathematical relation. Descriptions are proposed using the complex terminology of Helmholtz-Ellis and classical Greek mathematical theories of pitch relations [1]. This process relies upon a polyphonic pitch detection algorithm in order to provide fundamental pitches and their magnitudes over time.

This process is simply graphed in figure 5.1. In this method a best guess dominant frequency (f0) is determined for each point in time and these are compared to reference frequencies to determine their relationships. Highest ranking frequencies are determined by their relative magnitudes, higher magnitudes representing higher rank.

[1] See chapter 3

Stellenbosch University https://scholar.sun.ac.za

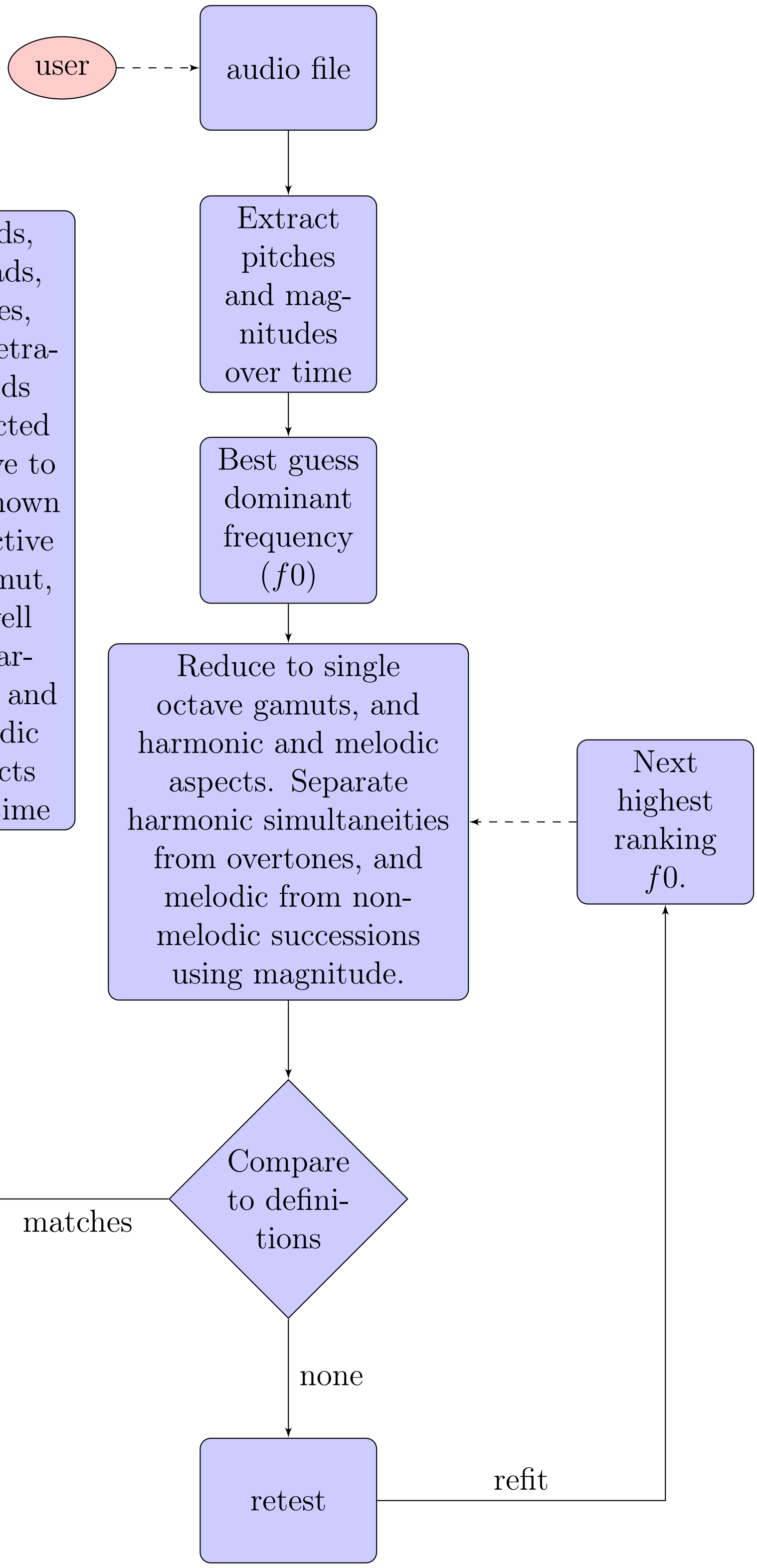

Figure 5.1: Simple Procedural Chart

Stellenbosch University https://scholar.sun.ac.za

In order to assist the rapid prototyping of such a software model it was found necessary to proceed with various converging developments, audio processing to return frequency numbers and pitch magnitudes, and the development of dictionary definitions for comparison. From these it was useful to apply limitations to magnitude, excluding data below a set magnitude thresholds, and pitch, reducing all pitches to a central octave derived from a best-guess tonic generator, and translating dictionary defined reference datasets to this same octave for comparison.

Various aspects of this proposed analysis have been prototyped, see Appendix A, and a sample analysis and its results constitute Appendix B.

5.1.1 Fundamental Pitch and Generators

In order to analyse the pitch relations of tonal music it is necessary that some method should be sought in order to provide definitive information regarding the exact fundamental tonal generator of any tonal sample undergoing analysis. This has proven a difficult task, as various instruments permit of widely variant and flexible pitching. For fixed pitch instruments a certain and definitive tonic may indeed be found by software analysis[2], however in music performed upon instruments with few or no fixed pitches, such as the voice or unfretted string instruments a great deal of variety may be found regarding exact pitch placement [3]. Such complications, relating not only to issues of timbre but also to those of tone production, were anticipated and discovered during the progress of this research. This has also introduced the need for a reappraisal of the idea of tonic generator as necessarily being a fixed pitch location.

The application of certain limitations in terms of magnitude and pitch range have proved themselves to be critical variables in achieving discreet and reflective results in the course of this investigation of tonal pitch phenomena. The need for these limits is due not only to timbral phenomena but also to other nuances of articulation and tone production.

5.1.2 Dictionary Definitions

Software definitions corresponding to the complex mathematical descriptions of tonal phenomena are here proposed. These furnish accurate datasets describing mathematically precise dictionaries of tonal relations, and terminology for defining melodic intervals and successions, harmonic compounds of pitch, and also non-melodic adjustments, deviations and successions.

Software definitions corresponding to these designs have been prototyped from the relation of two python modules, namely the *tuning* module of the Bregman Toolkit, and the *ratio* module designed by William S. Annis. These

[2] As it is in the analysis of the midi wave file, sample 3 in Appendix B.

[3] As may be seen in the violin wave file analysis, samples 1 and 2 in Appendix B

Stellenbosch University https://scholar.sun.ac.za

have been extended by subsequent development to include a greater variety of interval generation schemes and descriptions[4].

These dictionary definitions cover certain different facets of related tonal nature, namely, gamuts of tetrachords and octave scales, melodic successions as intervals, and harmonic chords as compounds of intervals[5]. These definitions furnish an explicitly defined library of simple intervals, as well as chords and scales, in great integral and enharmonic detail. This library is capable of allowing for automated cross-referencing and the retrieval of musically-meaningful practical information regarding the many contexts of pitch relations.These definitions are most succinctly presented by their identities within an octave gamut. Further this single octave gamut may be expanded to a multi-octave gamut and overlaid upon the extracted melodic curves and harmonic densities. Simple octave dictionaries may thus be made to serve not only to identify intervals reduced to a single octave-gamut, but also to identify melodic successions and harmonic densities throughout a larger gamut[6].

Further description of these dictionaries and their software implementation is presented in section 5.2.3.

5.1.3 Graphic Data and Logs

The prototyping presented in this research has used Python in Canopy Enthought (See figure 5.2), making use of its interactive console and text logging to assist debugging and testing. Primarily, the development of these dictionary definitions and software methods has proceeded using output information in the form of text logs representing the sample data.

This method of information retrieval lends well to a graphic illustration of scale usage. Information describing pitch, reduced to a central octave, being careful to preserve data regarding pitch occurrences and amplitudes, may prove useful to defining analytically useful octave-gamut sampling of intervals and attendant tetrachords and tonalities. Further, melodic curves and harmonic densities, related to the gamut sampling, but without the single octave restrictions, may be exhibited on a regular orthogonal pitch-time graph, or tabulated as samples.

From such data, series of simple data graphs of orthogonal relations may be easily plotted against each other. Consideration has been given, in modelling these graphs and logs, to the need for generating accurate general summaries regarding general pitch usage, represented as octave-gamuts, as well as general melodic and harmonic elements.

The description of melodic and harmonic elements requires further investigation to furnish a complete scheme of mathematical definitions and interval

[4] See section A.1.

[5] See extracts from these dictionary definitions in Appendix A.

[6] See the analysis prototyped in Appendix B.

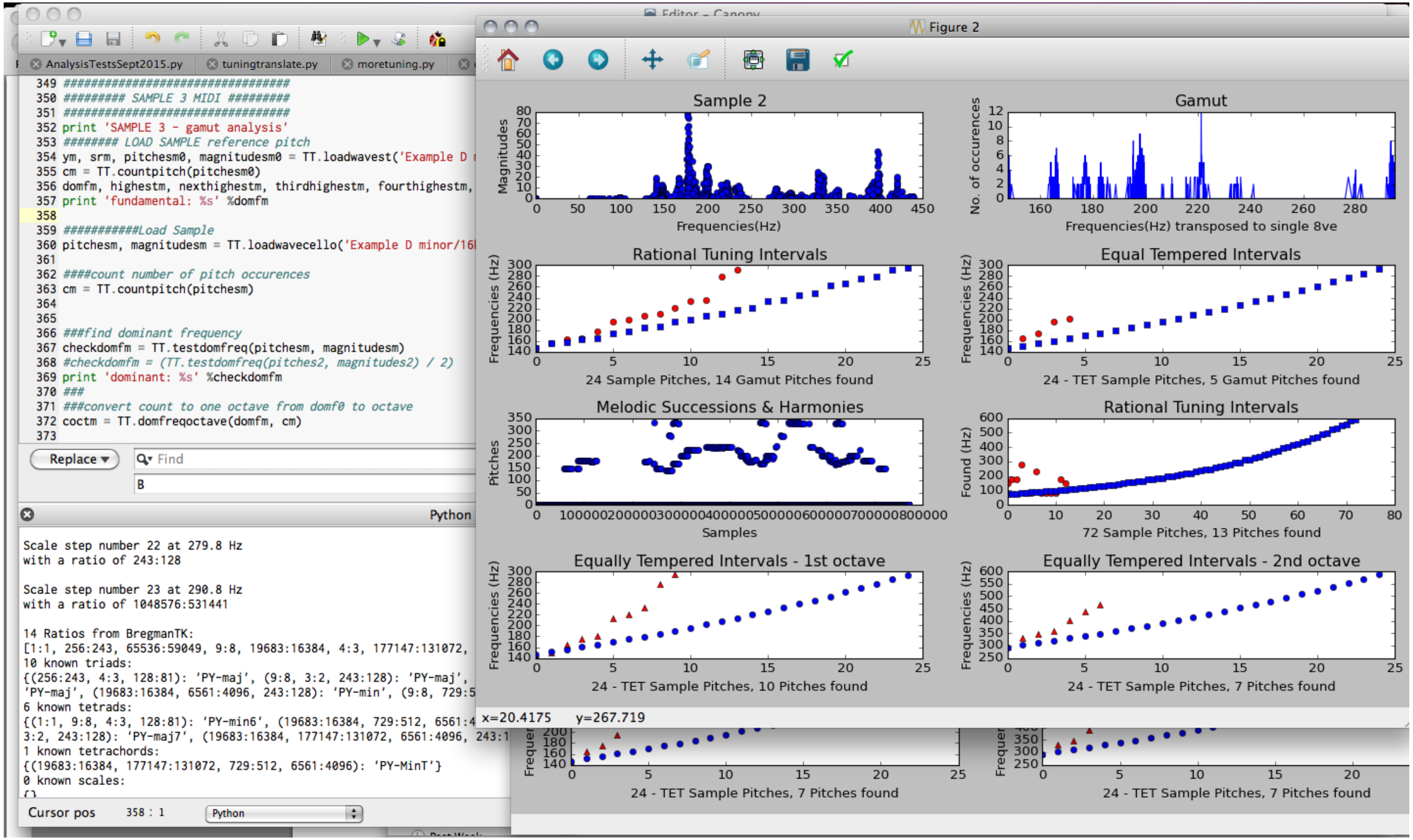

Figure 5.2: Screenshot showing example analysis and script output in Canopy.

Stellenbosch University https://scholar.sun.ac.za

distinctions. Additionally, the thorough definition of the many varieties of deviations, ornamentations and other non-melodic / non-harmonic adjustments and successions will require further development. That these last mentioned defintions are well within the scope of this research is exhibited by the wide variety of close intervals that are included in these dictionaries, many of whose mathematical descriptions are as yet unclassified by terminological definition.

The still ongoing development of these methods and their resultant analyses offer extensive scope for the hierarchic expression of pitch datasets. In addition to the raw ordering of pitch class intervals, as shown in the dictionaries, various related layers of analysis may be extracted. Interval distinctions represented using the gamut of an octave[7], may be expanded outwards applying the law of the octave and correlated against wider ranging melodic and harmonic data. See fig 5.3 for a simple diagram of this feature extraction procedure.

At present, the chief limitation to the effectiveness of this second layer of analysis lies in the current terminological definitions being limited to the intervals of an octave, this for both clarity and brevity in regarding the fundamental principals of this method of analysis and its still ongoing development. As such the relation of more specific melodic and harmonic data requires some extension and adaption to the definitions and methods developed by this preliminary research.

5.2 Python

The Scipy stack in conjunction with libraries such as the Bregman Toolkit and LibROSA have been used in developing and prototyping some of the methods outlined above. Using available feature extraction methods, and reference models, in the form of definitions of various intonations and temperament schemes, and developing these further where necessary, this research builds upon existing open-source code and contributes its development to the open-source development community.

5.2.1 Analysing Audio Data

LibROSA and the Bregman Toolkit offer a comprehensive assortment of feature extractors, from beat detection and tempo estimation to complex spectral analysis. These toolkits offer convenient and easy methods of accessing audio data, and allow for a variety of formats.

This research will deal here with the simple extraction of audio features from uncompressed mono wave data. LibROSA offers methods for summing

[7] In which the data are transposed to a central octave and may be analysed in two forms, 'gross' and 'net', where gross represents the pitch relations analysed prior to limitation, and net represents those pitches after limitation. Limitation amounting here to restrictions of magnitude and pitch range.

Stellenbosch University https://scholar.sun.ac.za

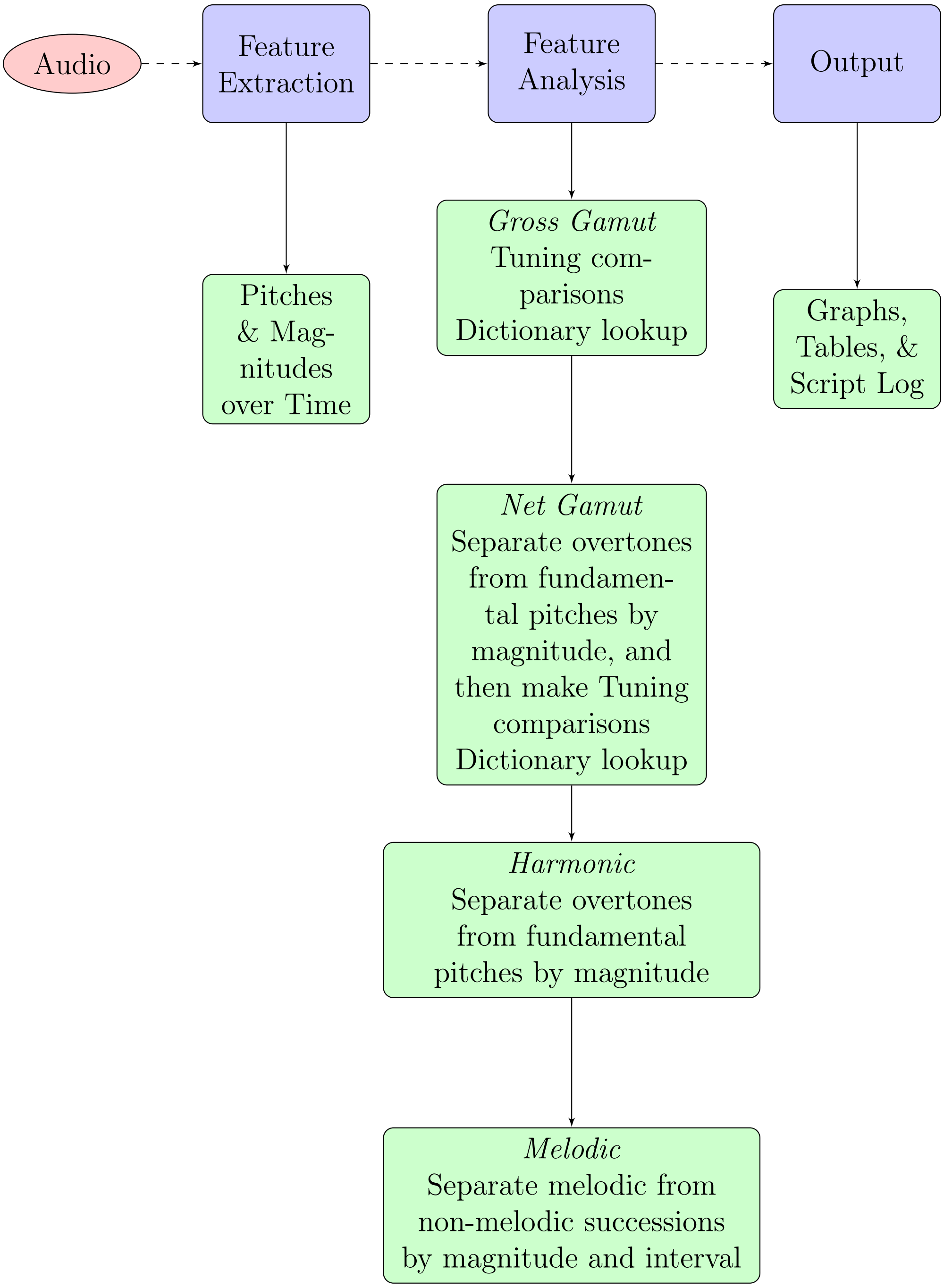

Figure 5.3: Feature Extraction and Analysis Workflow

Stellenbosch University https://scholar.sun.ac.za

```
#read n number of frames
wave.open('gmin.wav', 'r').readframes(n)
```

Figure 5.4: Matplotlib Example

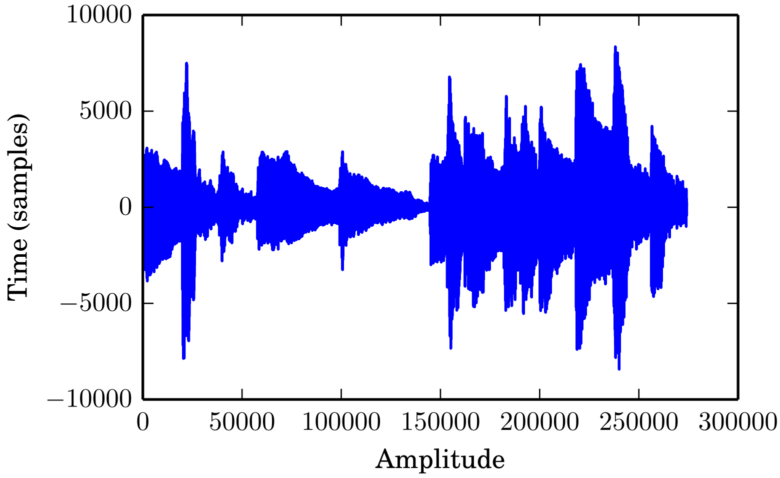

Figure 5.5: Matplotlib Example Graph

stereo files as mono data, and also methods for extracting separated harmonic and percussive content from the audio sample. This last is useful for generating independent beat and pitch data, extracting percussive elements from a harmonic analysis and vice versa.

5.2.2 Plotting with MatPlotLib

The MatPlot Library(MatPlotLib), containing *pyplot* for the Python environment, is a powerful tool for the rapid visualisation of data. This offers capabilities for complex graphing and plotting of the relations of audio data to the mathematical context dictionaries proposed by this research. See fig 5.4 and fig 5.5 for a simple example of this plotting capability, more extensive examples are also included in Appendix B.

In PythonTex[8], the LaTeXmodule used for the formatting of these python examples, Matplotlib (The python port of the MatLab Plotting Library.) is interacted with slightly differently. A slightly different code snippet will generate the same graph within LaTeX, as seen in fig 5.6.

[8] `https://github.com/gpoore/pythontex`

Stellenbosch University https://scholar.sun.ac.za

```
import scipy.io.wavfile as wavfile
wavfile.read('gmin.wav')
```

Figure 5.6: PythonTex Example

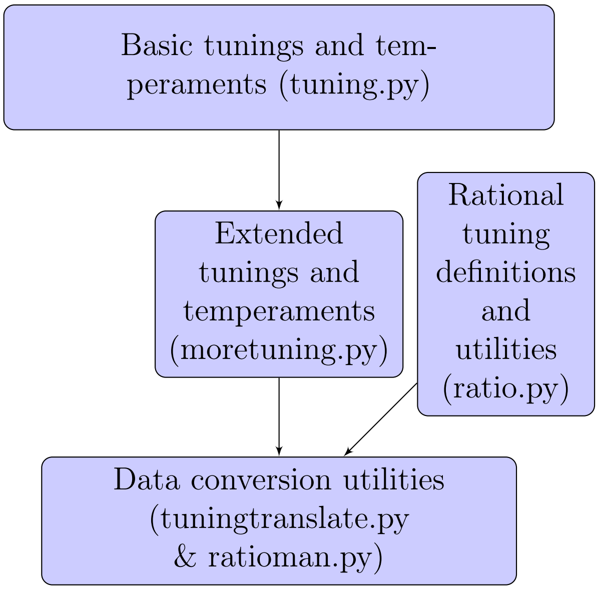

Figure 5.7: Reference Pitch Sets

5.2.3 Tuning References and Dictionary Definitions

Contained within the Bregman TK is a script dealing with various tuning methodologies. A standard approach to generating 12 scale steps from Pythagorean tuning, Equal Temperament, and an overtone-based Just intonation is described by this script. This research enlarges upon these approaches to generate scales with more than 12 steps using the same principles underlying these scripts definition of Equal temperament and Pythagorean Tuning. These extensions range from 25 notes to many hundreds of interval ratios, adding similar methods for generating scale steps describing Ptolemaic (5-limit) Augmented and Diminished systems. Methods have been developed for describing further complications of Pythagorean tuning arising from the multiplication of all its intervals by each other in turn. Further complications of these systems arise from considerations of interval sets related by commas, $^{81}/_{80}$, and the lesser and greater dieses, $^{128}/_{125}$ and $^{648}/_{625}$ respectively, as well as the multiplication of all of these various interval sets with themselves and each other. The basic structure of these reference pitch sets and their code development are outlined in figure 5.7 and further detailed in figure 5.8.

Stellenbosch University https://scholar.sun.ac.za

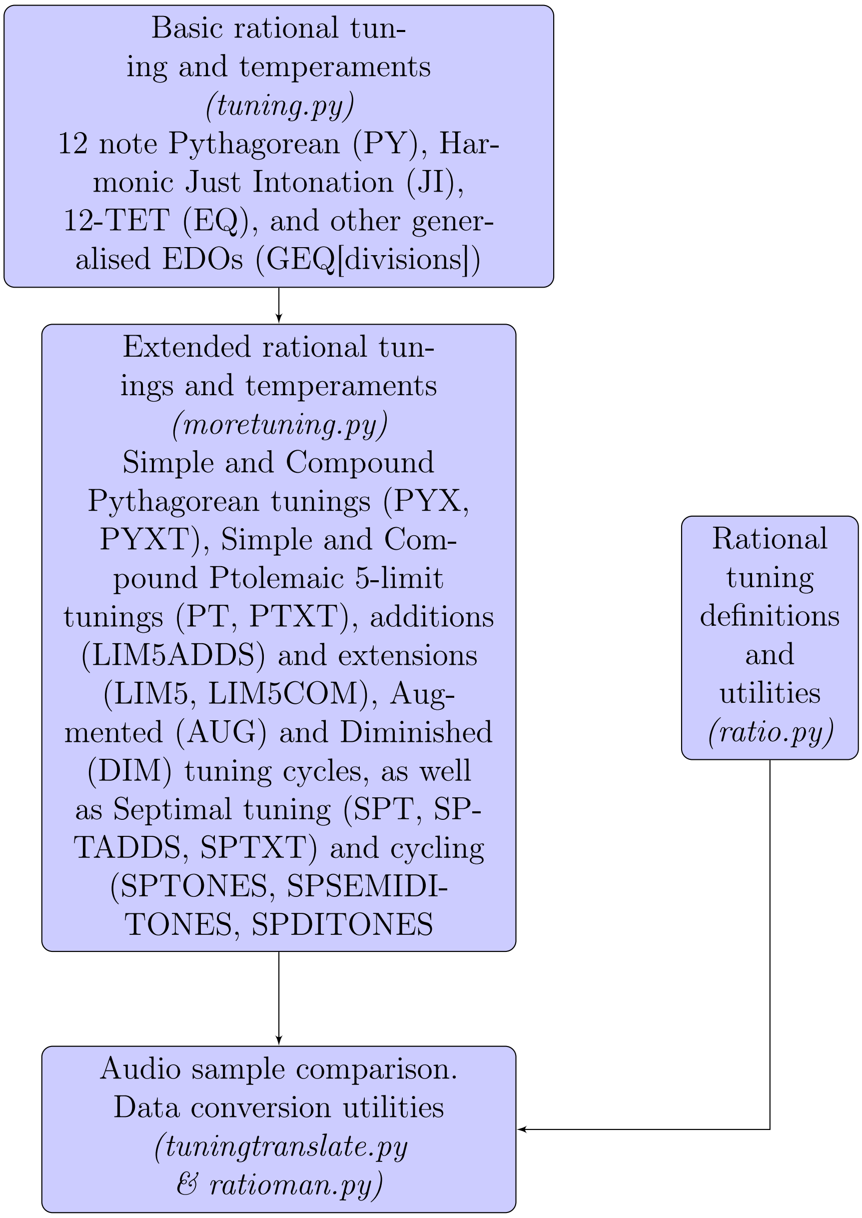

Figure 5.8: Detail of Reference Pitch Sets

Stellenbosch University https://scholar.sun.ac.za

These extensions are simply defined by the scripts *moretuning.py*[9] and *ratioman.py* adapted from the *tuning.py* script of the BregmanTK. In summary, these extensions are as follows.

1. *PythagoreanX* tuning is defined as a 25 tone extension of the 12 tone Pythagorean tuning initially offered by the BregmanTK[10]. These may also be multiplied against each other in turn resulting in the *Extended Pythagorean Tuning*[11].

2. *Ptolemaic* tuning is defined here as a set adding a limited set of certain 5-limit, comma related, extensions to the 12 tone Pythagorean tuning[12].

3. *Extended Ptolemaic* tuning is defined as the comma related 5-limit extension of the Extended Pythagorean tuning[13].

4. *Augmented*[14] tuning offers a cycle of intervals based upon ascending major thirds of $^5/_4$[15].

5. *Diminished*[16] tuning offers a cycle of intervals based upon ascending minor thirds of $^6/_5$[17].

6. *5-limit* tuning defines a set adding those intervals arising from cross multiplication of the extended Ptolemaic tuning set[18].

7. *5-limit additions* defines those intervals in 5-limit tuning that are not defined by the extended Pythagorean tuning set[19].

---

[9] See section A.1.1

[10] See figure A.1.

[11] See figure A.2.

[12] See figure A.15.

[13] See figures A.3 and A.16.

[14] Cycling of major thirds leads to a discrepancy between just theory and 12-TET symmetrical theory in regarding augmented intervals. In 12-TET theory the movement of three ditones/major thirds provides an exact octave. In 'just' theory the movement of three ditones, of $^5/_4$, leads to a very raised seventh (alternately a somewhat flattened octave), short of the octave by the lesser diesis of $^{128}/_{125}$.

[15] See figure A.4.

[16] Cycling of minor thirds leads to another notable discrepancy. In 12-TET theory the movement of four semiditones/minor thirds provides an exact octave. In 'just' theory the movement of four semiditones, of $^6/_5$, leads to a very flat minor ninth (alternately a rather raised octave), exceeding the octave by a greater diesis of $^{648}/_{625}$.

[17] See figure A.5.

[18] See figure A.17.

[19] See figure A.19.

Stellenbosch University https://scholar.sun.ac.za

8. The *Complete 5-limit* tuning defines a thorough listing of all possible intervals achieved by cross-multiplication of the intervals contained in the 5-limit tuning[20].

9. Certain *Septimal* tuning schemes are also modelled simply, using the cycling of the septimal tone $^8/_7$ and semiditone $^7/_6$[21]. These may be further added to by taking sets related to the aforementioned categories by the septimal comma $^{49}/_{48}$and dieses $^{28}/_{27}$, as well as by the cyclin gof other septimal intervals, such as the ditone $^9/_7$.

These scripts utilise the methods of the BregmanTK to create sets of fractions representing these respective tuning systems and their complications.

The BregmanTK contains methods for generating frequency numbers from these fractions, and allowing for the simple comparison of these reference intonations to the sampled data represented as frequency numbers.

The intersections of these comparisons may be referred to the ratio dictionaries described by the python script *ratio.py*[22]. This script provides simple and efficient methods for describing pitch relations in cents, tabling ratios, looking up dictionary entries of triads, and tetrads and other assorted function calls. This research has developed these functions into further tetrachord and scale dictionaries and attendant lookup functions, and also developed methods for translation between the ratios of *ratio.py* and the fractions given by the BregmanTK analysis functions. A script *tuningtranslate.py*[23] provides functions developed for the translation of data between these scripts varying output.

A script entitled *JustIntonation.py*[24] has also been rewritten in the course of this research. It is envisioned that this script may offer some additional resources to the ends of software description of simple dyads, or intervals of two related pitches. This script also offers a variety of operations for the manipulation and presentation of such intervals.

A hierarchic ordering of tuning and temperament ranking is described, in fig 5.9, based on the data presented in Chapter 3. In this scheme, tunings and temperaments are presented as higher ranking the closer they accord to mathematically simple integer representation, that is, the lower the prime n-limit or n-TET governing their distributions. Arithmetically mean distributions of tuning schemes are ranked higher than temperaments of logarithmically mean distributions. This is due to the facts of the reinforcing

[20] See figure A.20.

[21] See figures figures A.6, A.7, and A.8.

[22] See section A.1.2.

[23] See section A.1.4.

[24] An anonymous script, found at `http://www.snip2code.com/Snippet/173066/Just-intonation-experiments-in-Python/`, contributed by a user with the screen-name Endolith.

Stellenbosch University https://scholar.sun.ac.za

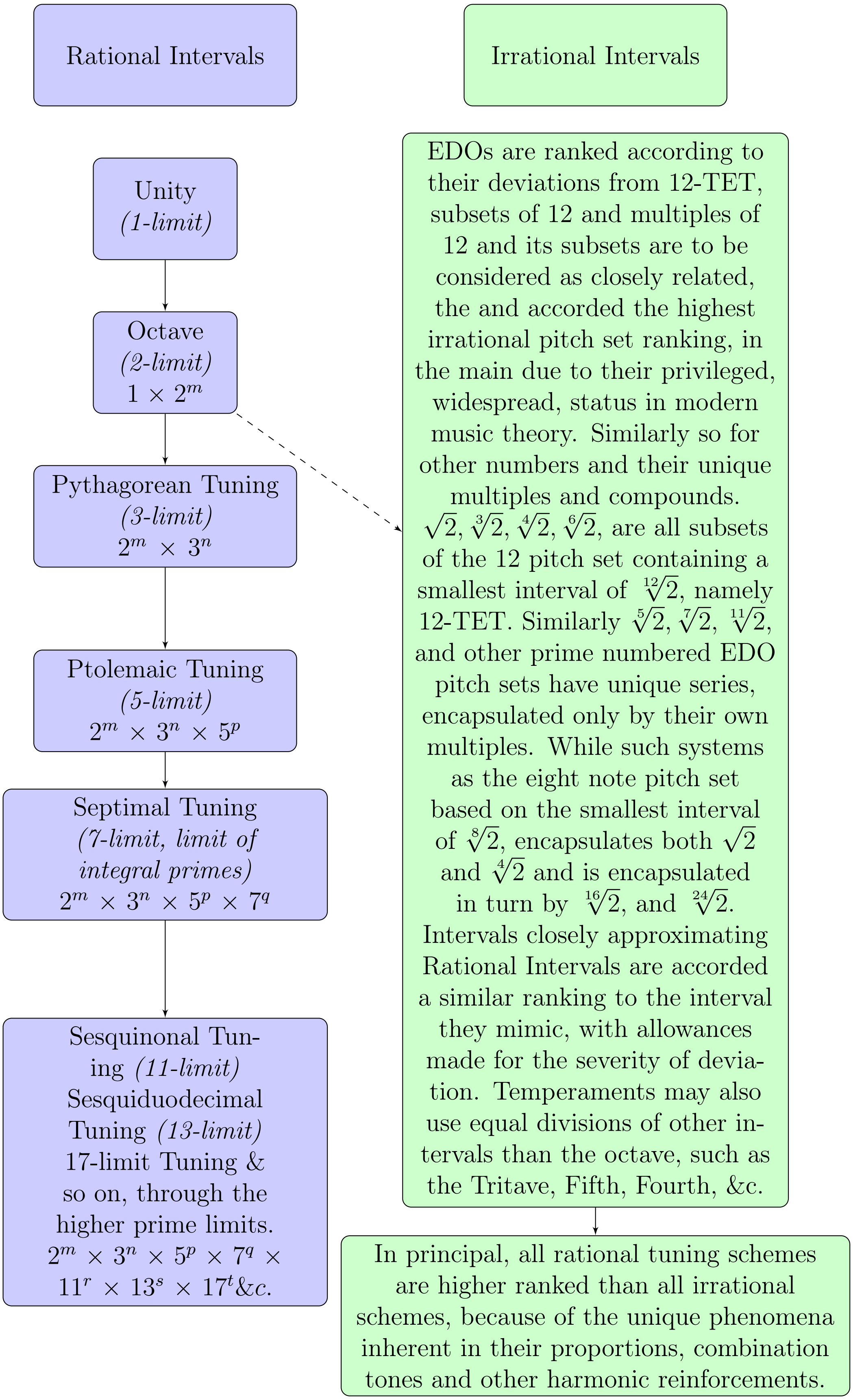

Figure 5.9: Hierarchies of Tuning and Temperament

Stellenbosch University https://scholar.sun.ac.za

sympathetic resonances that arise out of arithmetic integer and polynomial relations (Helmholtz, 1895:322).

5.3 Thresholding

The pitch values in hertz that are output by the feature extraction methods proposed in this model of software analysis are detailed to many decimal places. This detail requires some further differentiation relating enharmonic equivalents by some small interval threshold and in the limiting of minimum and maximum pitch range. This may prove particularly useful when deciding on the tonic pitch to use for interval relation.

This involves some shaping of data as frequency groupings. In determining the relative cent relations of extracted frequencies, and distributing them into fundamental and non-fundamental frequencies, it may prove practical to gauge the centre of tonal dominance, the key centre, in terms of a range of pitch locations, as opposed to a single exact pitch location.

Just Noticeable Difference (JND) conventions state that the ear cannot readily discern differences of less than 1.7 cents, and that different thresholds are valid for different frequencies at different loudness levels[25]. A closeness of around 2 cents, acute or grave, to a discrete pitch location may be safely gauged to be the relative distance a mathematically stated pitch interval may differ before it 'becomes', to the hearer at least, a different discernible pitch location. It will be useful to use some similar measure of thresholding in grouping the results of pitch extraction for analysis and definition.

This may also be successful with an even smaller threshold, such as one cent per interval class. 1200 intervals to the octave providing ample scaling with which to frame an in depth analysis and threshold correlations of intersections of tuning schemes.

This process would render intervals arising from rational tunings and irrational temperament schemes grouped within a range by some suitable threshold. This range would take the identity of the highest ranking enharmonic identity within the range. A readout of this gamut would exhibit the close similarities of such intervals and offer a scheme for the relative substitution of scale degrees between logarithmic and arithmetic distributions of pitch. These pitch class groupings and fine discriminations of their exact pitches would be useful for both informing compositional practices and performance and practical pitch theory.

[25]See JND glossary entry, and Newman (1933:173-174).

Stellenbosch University https://scholar.sun.ac.za

Chapter 6

Findings and Recommendations, Further Issues Suggested by this Study, and Conclusion,

I have elsewhere pointed out that the difficulty of defining tonality is the difficulty of describing any sensation whatever.

– (Tovey, 1973:7)

6.1 Findings

THIS research has framed a complex methodology in which a mathematical relational scheme of pitch analysis and corresponding terminological dictionaries are used to inform the interpretation of pitch extraction by audio software analysis. This presents an interpretation of musical pitch relations, and attendant scripting modules, in a form which may be of use to both musical researchers and musical practitioners.

This mathematical exploration of pitch space shows some promise for the elucidation of pitch practices, regarding scale gamut, harmonic and melodic usage, and other tonal pitch phenomena. However, there are a number of issues which have arisen and deserve further consideration.

At present, the prototyped software design[1] is capable of executing a limited mathematical analysis and tonal description. Testing has indicated the need for the introduction of thresholding parameters for pitch relations.

The mathematical description of pitch space relies upon the identification of a single identity representing unity, or $^1/_1$. Musical pitch usage being highly

[1] See Appendices A and B

Stellenbosch University https://scholar.sun.ac.za

flexible, as shown by the considerable variation of closely related enharmonic pitches extracted by software analysis, the pitch location representing unity often shows considerable fluctuations of exact pitch number.

The use of limitations to magnitude and pitch range have proved themselves to be critical variables in achieving discreet and reflective results in the course of this prototyping this methodology of pitch analysis. The need for these limits is due not only to timbral phenomena but also to other nuances of articulation and tone production[2]. A threshold parameter, serving to group closely related pitch instances into related pitch classes, may be useful to meaningfully relating the enharmonic equivalents of the various orders of tunings and temperaments developed in this research.

The design of pitch extraction, prior to submission for automated mathematical and terminological analysis, requires further development, both in regards to the equation of enharmonic identities by some variable threshold[3] and methods for the notation of such saliences of pitch deviations.

More advanced analysis of modulation and expressive intonation may partake of useful description by treating of a modulating unity. Aspects of tonal phenomena such as modulations of tonic deserve special investigation for potential methods of modelling the complexity of relations they introduce to such mathematical analyses of pitch as this research proposes. Software methods for defining modulating phenomena, such as the iterative generation of modulating reference sets using machine learning methods, may prove useful to further development.

6.2 Further Issues Suggested by this Study

The integration of SOM[4] integration in further development of this software modelling may allow for the automation of a machine learning structure capable of hierarchically ordering tuning and temperament analysis and also for iterative reference generation as describing above. Further this may prove capable of timbral recognition [5].

The terminological definition, presently confined to the range of an octave, excluding octave equivalences of intervals, require further extension to the multi-octave ranges required for the analysis of melodic and harmonic tonal phenomena. It is further desirable that this should by some means express the relations of octave equivalences, as well as other harmonic consonances. It may be desirable to use machine learning methods of feedback to extend

[2] See section 5.1.1.

[3] See section 5.3.

[4] See section 4.3.4.2.

[5] See section 5.1.1.

Stellenbosch University https://scholar.sun.ac.za

ranges of reference pitch identities by iteration from the extracted range of the sampled data.

The methods generated in this research may also be made capable of analysing real-time audio[6]. A unique method for analysing pitch and deciding tonic pitch would need to be found for this approach.

Investigations into the timbral utilities of hierarchical categorisation of tuning and temperament schemes may prove further useful to the mathematical classification of timbre, and allow for the discerning of individual melodic lines, and their instrument type, in complex samples.

Investigation into how these mathematical relations may be used to perform rhythmic and metric analysis after the *poetics*[7] of classical Greeks. LibRosa provides methods for beat tracking, tempo / bpm estimation, also containing utilities allowing for the separation of a signal into harmonic (pitch) and percussive (regularity of magnitudes) component arrays. This beat tracking facility may be useful to further developing the results of this research into a mathematical description of rhythmic categorisation.

The scripts and functions described in this research, once suitably developed, may be integrated into a software application for the mathematical analysis of recorded audio using gamut analyses of tetrachords and scales, melodic interval recognition, and harmonic discrimination.

6.3 Development of Software Libraries

Further development of these scripts may make a useful script tool-kit for musicologists. Further developments of threshold parameters[8] would make it a useful tool for musicologists, ethnographers, acousticians and musicians alike.

The BregmanTK extension scripts developed in this research have been submitted to their developers at Dartmouth's Bregman Labs and are currently being considered for inclusion into further releases of the BregmanTK. William S. Annis, the author of the *ratio.py* script extended by this research, has also expressed his appreciation of the extensions. These scripts are included in this research and will be further elaborated in an open-source release.

Python, as a highly extensible software framework, was deliberately chosen for its ease of coding and the great wealth of open-source development.

As in the case of the extracted fundamental pitches and their reduction to data arrays, similar data may be gathered and arrayed from the remaining harmonics of the overtone spectrum. Treatment of these overtone spectra has been dispensed with for the sake of this study, however such analysis may prove

[6] Using the pyaudio module, available at `http://people.csail.mit.edu/hubert/pyaudio/`.

[7] See Aristotles Poetics, Aristoxenus

[8] See section **??**.

Stellenbosch University https://scholar.sun.ac.za

useful in expanding the scope of this approach into the realms of timbral analysis, and investigation of the occurrences of differential tones, summational tones and other combinational tonal phenomena attributed to tonal wave interferences. Further possible extensions of these methods of data analysis may allow for the automated recognition and mathematically rigourous classification of other musical information, such as melodic patterns greater than simple successions, phrase recognition, theme and variation recognition, and stress / tempo and modulation plotting.

6.4 Conclusion

The modeling of the many possibilities of tonal context rely for their distinction upon the potential for discerning and measuring the fundamental pitch, or pitches, exhibited by a sound, or sounds, over time. Scale gamut, melodic and harmonic descriptions of sequential and simultaneous pitch elements, as well as more subtle non-melodic/non-harmonic elements of close successions and deviations of pitch, together with other tonal relations, such as combination tones and overtones may be extracted by analysis from the accurate registration of pitch heights, durations and magnitudes.

Data arrays representing fundamental pitch, including amplitude, timing duration, and sequencing, allow for the extraction of analytically useful information regarding pitch ranges and intervals defined mathematically. These identities are then capable of being related to musically descriptive terminology drawn from mathematically explicit dictionaries of theoretic tonal relations[9]. In order to describe these datasets, information indicating a dominant reference pitch, or key-centre, must be extracted. This dominant pitch, treated as mathematical unity, allows for the relational mapping of intervals, chordal movements, melodic successions and other tonal phenomena. These are expressed by relation to pitch ranges defined in frequencies, log cents and ratios.

As the mathematical model here employed requires some critical junction upon which to base the tonal radix point for its analytic purposes, it is at this early stage mainly applicable to instances of distinctly tonal music, that is music having some fixed, or static tonic generator. In further cases of analysis involving atonal music, and indeed this must include much modern classical and jazz music and any other modulatory music, further methods must be sought for deriving, and, if necessary, for correcting the analysis of music in order to correspond to the relations to a dynamic, or moving, tonic generator.

The software method modelled in this research, based on historic schemes of mathematical pitch definition, has resulted in the practical development of

[9] In the case of these sample dictionaries, the terminology has been drawn from the classical Greek theory, which offers the benefit of having precise terms for the classifications of unique mathematical intervals. See sections 2.1.2, 2.1.3 3.3, and 5.1.2.

Stellenbosch University https://scholar.sun.ac.za

tuning and temperament definitions describing a wide variety of mathematically defined pitch identities. These have been ordered in a hierarchy proposed according to their mathematical means.

While there are further issues resulting from this methodology[10], these were mostly anticipated and methods of further refinement in regards to these issues has been proposed. The results of the prototype analysis described in appendix B indicate that it may be worthwhile to investigate possibilities for filtering analysis according to instrument specific 'timbral-sieves'. This may, within certain limits, also provide some way to differentiate voices and their instrumentation within a complex audio mix. The use of limiting factors, such as those groupings prescribed by theories of just noticeable differences (JND), may be considered and dealt with by incorporating such variations and their approximations in additional analysis, informed by the data furnished by the initial analysis.

In view of the difficulties encountered by this research regarding the tracking of the fundamental pitch of sound over time[11], certain aspects of the MPM[12], notably the application of the gaussian function, may prove useful to a more robust design of fundamental pitch tracking. The introduction of a threshold relating extracted pitch to acceptable pitch relations, as described by the methods of the MPM algorithm threshold[13] may go some ways towards offering a more acceptable method for grouping approximating pitch distributions. It would be most useful for the extension of this preliminary research to investigate ways to incorporate aspects of this approach in analogous methods utilising Python.

It may be argued that the notion of pitch ranges, as opposed to fixed pitch points, may better serve the analysis which this research suggests. This also offers a theoretical similarity to the descriptions given by Aristoxenus and other classical Greek theorists of the movements of sounds in their ranges, typified by the practical technique of finger emplacement. [14] In order to pursue such relations of fixed data to ranges, it will be further necessary to determine some acceptable method of both determining and allocating such ranges, these being fitted to various resolutions. At present, the data furnished by the prototype analysis results in the representation of fixed pitch points. These have been rounded to one decimal place in order to represent some measure of this desired approximation, however this aspect is deserving of further development in order to group close pitches in their categoric ranges and approximations to strictly defined mathematical interval relations.

---

[10]See section 6.2.

[11]See section 5.1.1

[12]McLeod Pitch Method

[13]See section 4.3.2.2.

[14]For more information regarding the notion of *topos*, or a range of moveable notes, see Aristoxenus in Barker (1989:141-142).

Stellenbosch University https://scholar.sun.ac.za

The software prototype described in appendix A and the analysis described in appendix B have shown themselves to offer useful methods of pitch analysis. While proving this methods potential for the relation of pitch phenomena to mathematical statement and historic practices rooted in such quantitative methods, the outcome of this preliminary investigation requires some further development before it can be declared a satisfactorily robust method for pitch analysis. The numerous improvements indicated above, arising from consideration of the results obtained by this preliminary investigation, are satisfying results, indicating fertile avenues for investigation into the explication of mathematical representations of tunings and temperaments and their representations as perceived pitch relations in music theory.

Stellenbosch University https://scholar.sun.ac.za

Appendices

Stellenbosch University https://scholar.sun.ac.za

Appendix A

Appendix A: Python Excerpts

A.1 Python Modules

Code excerpts from the Python modules created in the course of this research are here presented with a brief elaboration of their utilities.

A.1.1 moretuning.py

This module was developed using methods contained in the tuning module of the Bregman Toolkit. These have been considerably reworked and used to generate many varieties of tuning not provided in the Bregman Toolkit. Some of these are described below.

The code in figure A.1 defines a tuning class extending the definitions for 3-limit Pythagorean temperament, as fractions, from the 12 ratios given in the tuning module of the Bregman Toolkit to a 24 ratio interval set.

The code in figure A.2 multiplies the 24 ratio interval set above by themselves and reduces these values to relations within a single octave gamut.

The code in figure A.3 multiplies the 24 ratio interval set above by a comma grave and acute to produce the relations within a single octave gamut corresponding to the 5-limit complication of 3-limit Pythagorean Tuning. This has here been termed Ptolemaic tuning.

The code in figure A.4 and figure A.5 exhibit the ratios of 5-limit Augmented and Diminished relations.

The code in figures A.6, A.7, and A.8 exhibit similar cycles of some Septimal, or 7-limit, interval relations.

A.1.2 ratio.py

The ratio module is adapted from a python module written by William S. Annis[1]. This module provides a musical ratio formatting that further allows

[1] `http://www.lingweenie/site/music/RCS/ratio.py.txt`

Stellenbosch University https://scholar.sun.ac.za

```
def PythagoreanX(self, N=12):
        """
            Generate 12 steps of PythagoreanX tuning as rational numbers
            from N cycles in P5 and P4 directions. Removed limitation of
            approximating EQ intervals.
        """
        ratios = []
        self.pitch_ratios_sharp = [0]*N
        self.pitch_ratios_flat = [0]*N
        for k in range(N):
            ratios.append(Fraction(3,2)**k)
            while ratios[k] > 2:
                ratios[k] /= 2
        for r in range(N): self.pitch_ratios_sharp[(r*7)%N] = ratios[r%12]

        ratios = []
        for k in range(1,N+1):
            ratios.append(Fraction(2,3)**k)
            while ratios[k-1] < 1:
                ratios[k-1] *= 2
        for r in range(N): self.pitch_ratios_flat[((r+1)*5)%N]=ratios[r]
        self.pitch_ratios_pythagoreanx = self.sort_ratios
                        (self.pitch_ratios_flat + self.pitch_ratios_sharp)
        return self.pitch_ratios_pythagoreanx
```

Figure A.1: 24 Pythagorean Tuning

for the retrieval of dictionary definitions based on the musical ratios of chords over a greatest common denominator. Tetrachord and scale dictionaries were adapted from the existing method for calling chords, as triads and tetrads, which was also extended and reworded. Also provides methods for splitting ratios into arithmetically equal parts, creating tables of musical Duodenes, tabling ratios, converting scale ratios to interval ratios, unpacking fractions to musical ratios, and determining cent differences.

Figure's A.9, A.10, and A.11 show some of the definitions available in this module, and figures A.12, A.13, and A.14 show the methods used to match these definitions to a list of ratios.

A.1.3 ratioman.py

This module `ratioman.py`adds further definitions to the Bregman tuning module and the above moretuning addition. Figure A.15 provides a 12 pitch set of Ptolemaic 5-limit intervals.

Stellenbosch University https://scholar.sun.ac.za

```
def PythagoreanExt(self):
        """   ######Extending Pythagorean Tuning####
        multiplying Pythagorean Tuning by each
        of it's own intervals in turn, excepting the octave
        """
        pt=[]
        for i in MT.PYX[0:24]:
            PyX = [Fraction(i)]*25 #make list from each interval
            PyX2 = np.prod([MT.PYX, PyX], axis=0) #multiply interval by scale
            pt.append(PyX2) #append array of new intervals to list PT
        pt.append(MT.PYX)
        PT = np.unique(np.concatenate(pt))

        octaves = []
        PTO = []
        for j in PT:
            if Fraction(j)*Fraction(1,2) in PT:  #put octaves in own list
                octaves.append(j)
            if Fraction(j)*Fraction(1,2) not in PT:#get single octave interval
                PTO.append(j)

        for p in PTO:   #convert higher octave intervals to single octave
            d = Fraction(p)*Fraction(1,2)
            if d > Fraction(1,1):
                if d < Fraction(2,1):
                    PTO.remove(p)
                    PTO.append(d)
                if d > Fraction(2,1):
                    e = d*Fraction(1,2)
                    if e < Fraction(2,1):
                        PTO.remove(p)
                        PTO.append(e)
                    if e > Fraction(2,1):
                        f = e*Fraction(1,2)
                        if f < Fraction(2,1):
                            PTO.remove(p)
                            PTO.append(f)
                        if f > Fraction(2,1):
                            g = f*Fraction(1,2)
                            if g < Fraction(2,1):
                                PTO.remove(p)
                                PTO.append(g)
        return PTO
```

Figure A.2: Extended Pythagorean Tuning

Stellenbosch University https://scholar.sun.ac.za

```python
def Ptolemaic(self, N=12):
    # code from PythagoreanX tuning goes here
        ptol = []   ##calculate comma acute
        PtolX = [Fraction(81, 80)]*len(self.pitch_ratios_pythagoreanxpt)
        PtolX2 = np.prod([self.pitch_ratios_pythagoreanxpt, PtolX], axis=0)
        ptol.append(PtolX2)
        PTOL = np.unique(np.concatenate(ptol))
        PTOLO = []
        PToloctaves = []
        for k in PTOL:
            if Fraction(k)*Fraction(1,2) in PTOL:
                PToloctaves.append(k)
            if Fraction(k)*Fraction(1,2) not in PTOL:
                PTOLO.append(k)
        for p in PTOLO:
            fu = Fraction(p)*Fraction(1,2)
            if fu > Fraction(1,1):
                PTOLO.remove(p)
                PTOLO.append(fu)
        ptola = []  ##calculate comma grave
        PtolY = [Fraction(80, 81)]*len(self.pitch_ratios_pythagoreanxpt)
        PtolY2 = np.prod([self.pitch_ratios_pythagoreanxpt, PtolY], axis=0)
        ptola.append(PtolY2)
        PTOLa = np.unique(np.concatenate(ptola))
        #print PTLa
        PTOLU = []
        PToluctaves = []
        for k in PTOLa:
            if Fraction(k)*Fraction(1,2) in PTOLa:
                PToluctaves.append(k)
            if Fraction(k)*Fraction(1,2) not in PTOLa:
                PTOLU.append(k)
        for p in PTOLU:
            fuu = Fraction(p)*Fraction(1,2)
            if fuu > Fraction(1,1):
                PTOLU.remove(p)
                PTOLU.append(fuu)
        for p in PTOLU:
            fu1 = Fraction(p)*Fraction(1,2)
            if fu1 > Fraction(1,1):
                PTOLU.remove(p)
                PTOLU.append(fu1)
        PtolemaicoX = np.unique(
                PTOLO + PTOLU + self.pitch_ratios_pythagoreanxpt)
        self.pitch_ratios_ptolemaicx = PtolemaicoX[1:len(PtolemaicoX)-1]
        return self.pitch_ratios_ptolemaicx   #less extrema commas
```


Figure A.3: Extended Ptolemaic Tuning

Stellenbosch University https://scholar.sun.ac.za

```
def Augmented(self, N=12):
        """
            Generate 12 steps of Just major thirds as rational numbers
            from N cycles in M3 and m6 directions.
        """
        ratios = []
        self.pitch_ratios_sharp = [0]*N
        self.pitch_ratios_flat = [0]*N
        for k in range(N):
            ratios.append(Fraction(5,4)**k)
            while ratios[k] > 2:
                ratios[k] /= 2
        for r in range(N): self.pitch_ratios_sharp[(r*7)%N] = ratios[r%12]

        ratios = []
        for k in range(1,N+1):
            ratios.append(Fraction(4,5)**k)
            while ratios[k-1] < 1:
                ratios[k-1] *= 2
        for r in range(N): self.pitch_ratios_flat[((r+1)*5)%N]=ratios[r]
        self.pitch_ratios_augmented = self.sort_ratios
                        (self.pitch_ratios_flat + self.pitch_ratios_sharp)
        return self.pitch_ratios_augmented
```

Figure A.4: 5-limit Augmented Cycles

Figure A.16 describes an Extended Ptolemaic tuning, using comma acute and grave extensions to the extended Pythagorean ratios.

The code straddling figures A.17 and A.18 returns a set of 152 5-limit ratios.

Figure A.19 returns a set of 127 5-limit ratios not present in the Pythagorean Extended tuning set.

Figure A.20 returns a set of 450 5-limit ratios.

Figure A.21 provides a simple method for generating float values from 0 to 1representing any particular n-tet temperament scheme.

Figure A.22 depict a series of sample reference pitch sets providing various tunings systems as frequencies from 440Hz to 880Hz.

Stellenbosch University https://scholar.sun.ac.za

```
def Diminished(self, N=12):
        """
            Generate 12 steps of Just minor thirds as rational numbers
            from N cycles in m3 and M6 directions.
        """
        ratios = []
        self.pitch_ratios_sharp = [0]*N
        self.pitch_ratios_flat = [0]*N
        for k in range(N):
            ratios.append(Fraction(6,5)**k)
            while ratios[k] > 2:
                ratios[k] /= 2
        for r in range(N): self.pitch_ratios_sharp[(r*7)%N] = ratios[r%12]

        ratios = []
        for k in range(1,N+1):
            ratios.append(Fraction(5,6)**k)
            while ratios[k-1] < 1:
                ratios[k-1] *= 2
        for r in range(N): self.pitch_ratios_flat[((r+1)*5)%N]=ratios[r]
        self.pitch_ratios_diminished = self.sort_ratios
                        (self.pitch_ratios_flat + self.pitch_ratios_sharp)
        return self.pitch_ratios_diminished
```

Figure A.5: 5-limit Diminished Cycles

Stellenbosch University https://scholar.sun.ac.za

```
def SeptimalTones(self, N=12):
        '''
            Generate 12 steps of cycles of 8:7 as rational numbers
            from N cycles.
        '''
        ratios = []
        self.pitch_ratios_sharp = [0]*N
        self.pitch_ratios_flat = [0]*N
        for k in range(N):
            ratios.append(Fraction(8,7)**k)
            while ratios[k] > 2:
                ratios[k] /= 2
        for r in range(N): self.pitch_ratios_sharp[(r*7)%N] = ratios[r%12]

        ratios = []
        for k in range(1,N+1):
            ratios.append(Fraction(7,8)**k)
            while ratios[k-1] < 1:
                ratios[k-1] *= 2
        for r in range(N): self.pitch_ratios_flat[((r+1)*5)%N]=ratios[r]
        self.pitch_ratios_septimaltones = self.sort_ratios
                        (self.pitch_ratios_flat + self.pitch_ratios_sharp)
        return self.pitch_ratios_septimaltones
```

Figure A.6: Septimal Tone Cycles

Stellenbosch University https://scholar.sun.ac.za

```python
def SeptimalSemiditones(self, N=12):
        '''
            Generate 12 steps of cycles of 7:6 as rational numbers
            from N cycles.
        '''
        ratios = []
        self.pitch_ratios_sharp = [0]*N
        self.pitch_ratios_flat = [0]*N
        for k in range(N):
            ratios.append(Fraction(7,6)**k)
            while ratios[k] > 2:
                ratios[k] /= 2
        for r in range(N): self.pitch_ratios_sharp[(r*7)%N] = ratios[r%12]

        ratios = []
        for k in range(1,N+1):
            ratios.append(Fraction(6,7)**k)
            while ratios[k-1] < 1:
                ratios[k-1] *= 2
        for r in range(N): self.pitch_ratios_flat[((r+1)*5)%N]=ratios[r]
        self.pitch_ratios_septimalsemiditones = self.sort_ratios
                        (self.pitch_ratios_flat + self.pitch_ratios_sharp)
        return self.pitch_ratios_septimalsemiditones
```

Figure A.7: Septimal Semiditone Cycles

Stellenbosch University https://scholar.sun.ac.za

```
def SeptimalDitones(self, N=12):
        '''
            Generate 12 steps of cycles of 9:7 as rational numbers
            from N cycles.
        '''
        ratios = []
        self.pitch_ratios_sharp = [0]*N
        self.pitch_ratios_flat = [0]*N
        for k in range(N):
            ratios.append(Fraction(9,7)**k)
            while ratios[k] > 2:
                ratios[k] /= 2
        for r in range(N): self.pitch_ratios_sharp[(r*7)%N] = ratios[r%12]

        ratios = []
        for k in range(1,N+1):
            ratios.append(Fraction(7,9)**k)
            while ratios[k-1] < 1:
                ratios[k-1] *= 2
        for r in range(N): self.pitch_ratios_flat[((r+1)*5)%N]=ratios[r]
        self.pitch_ratios_septimalditones = self.sort_ratios
                        (self.pitch_ratios_flat + self.pitch_ratios_sharp)
        return self.pitch_ratios_septimalditones
```

Figure A.8: Septimal Ditone Cycles

Stellenbosch University https://scholar.sun.ac.za

```
# PY = Pythagorean tuning,
# 3 limit base
# PT = Ptolemaic tuning
# 5 limit inclusions
# SPT = Septimal tuning
# 7-limit inclusions
CHORDS = {'PT-maj': (4, 5, 6),     #PT I IV V ###TRIADS
          'PT-min': (10, 12, 15), #PT vi ii iii
          'PT-aug': (16, 20, 25),   #PT +
          'PT-dimtriad': (25, 30, 36),   #PT vii

          'PY-maj': (64, 81, 96),
          'PY-min': (54, 64, 81),
          'PY-aug': (4092, 5184, 6561), #
          'PY-dimtriad': (729, 864, 1024),

          'SPT-dimtriad': (5, 6, 7),
          'SPT-minor': (6, 7, 9),
          'SPT-dom7omit5': (4, 5, 7),   # 1-3-7
          'SPT-dom7omit3': (4, 6, 7),   # 1-5-7

          'PT-augquartad':(64, 80, 100, 125),  #PT + b8ve   ###TETRADS
          'PT-dim': (125, 150, 180, 216), #PT o
          'PT-maj7': (8, 10, 12, 15),   #PT I IV
          'PT-min7': (10, 12, 15, 18),  #PT vi ii iii
          'PT-dom7': (20, 25, 30, 36),  #PT V
          'PT-m7b5': (25, 30, 36, 45),  #PT vii half dim
          'PT-min6': (30, 36, 45, 50),  #PT ii / vii m7b5 inv3B
          'PT-maj6': (12, 15, 18, 20),  #PT

          'PY-maj7': (128, 162, 192, 243),
          'PY-maj6': (64, 81, 96, 108),
          'PY-min7': (54, 64, 81, 144),
          'PY-min6': (432, 512, 648, 729),
          'PY-m7b5': (729, 864, 1024, 1296),

          'SPT-dom7': (4, 5, 6, 7),
          '11lim-min7': (6, 7, 9, 11),  #11-limit
          'SPT-dom9': (4, 5, 6, 7, 9),


          'PT-maj9': (8, 10, 12, 15, 18),     ###HEXADS
          'PT-min9': (20, 24, 30, 36, 45)
          }
```

Figure A.9: Chord Relations

Stellenbosch University https://scholar.sun.ac.za

```
TETRACHORDS = { 'PT-MajT': (24, 27, 30, 32), #greater tone leads
            'PT-majT': (36, 40, 45, 48), #lesser tone leads
            'PT-MinT': (120, 135, 144, 160), #greater tone leads
            'PT-minT': (45, 50, 54, 60), #lesser tone leads st=27:25
            'PT-phrygT': (15, 16, 18, 20),
            'PT-lydT': (32, 36, 40, 45),  #raised fourth
            'PT-asChromaT': (120, 128, 135, 160), #ascending
            'PT-desChromaT': (60, 72, 75, 80),    #descending
            'PT-harmChromaT': (60, 64, 75, 80), #'harmonic'

            'PY-MajT': (192, 216, 243, 256), #greater tones
            'PY-majT': (576, 640, 729, 768), #lesser tone incl.
            'PY-MinT':(216, 243, 256, 288), #greater tone in lead
            'PY-Mint':(40, 45, 48, 54), #greater tones 4th=27:20
            'PY-minT': (27, 30, 32, 36), #lesser tone in lead
            'PY-phrygT': (405, 432, 480, 540),

            'SPT-majT': (21, 24, 27, 28)  #Septimal
          }
```

Figure A.10: Tetrachord Relations

Stellenbosch University https://scholar.sun.ac.za

```
SCALES = {  'PT-MajorS': (24, 27, 30, 32, 36, 40, 45), #Ionian
            'PT-majorS': (120, 135, 150, 160, 180, 200, 216), #Mixo
            'PT-minorS': (120, 135, 144, 160, 180, 192, 216),  #Aeolian
            'PT-MinorS': (120, 135, 144, 160, 180, 200, 216),  #Dorian
            'PT-phrygS': (60, 64, 72, 80, 90, 96, 108),  #Phrygian
            'PT-locriS': (45, 48, 54, 60, 64, 72, 81),  #Locrian

            'AristEratEnharmonicS': (60, 76, 78, 80, 90, 114, 117),#Barkerp347
            'DidymusEnharmonicS': (48, 60, 62, 64, 72, 90, 93),
            'AristoxHemChromaticS': (120, 148, 154, 160, 180, 222, 231), #p348
            'AristoxEratTonChromaticS': (30, 36, 38, 40, 45, 54, 57),
            'AristoxSoftDiatonicS': (60, 72, 76, 80, 90, 105, 114), #p349
            'AristoxTenseDiatonicS': (30, 34, 38, 40, 45, 51, 57),
            'DidymusDiatonicS': (48, 54, 60, 64, 72, 81, 90),
            'EvenDiatonicS': (18, 20, 22, 24, 27, 30, 33), #p350
            'OvertoneS': (8, 9, 10, 11, 12, 13, 14, 15), #overtone series
            'OverChromS': (16, 17, 18, 19, 20, 21, 22, 23,
                            24, 25, 26, 27, 28, 29, 30, 31)#overtone chromatics
            }
```

Figure A.11: Scale Relations

Stellenbosch University https://scholar.sun.ac.za

```
def triads(scale):
    """Generate a list of known triads in a scale."""
    n = len(scale)
    chords = {}
    concords = CONCORDS.keys()
    for i in range(n):
        for j in range(i, n):
            for k in range(j, n):
                chord = gamut2identities([scale[i], scale[j], scale[k]])
                chord.sort()
                chord = tuple(chord)
                if chord in concords:
                    chords[(scale[i], scale[j], scale[k])] = CONCORDS[chord]

    return chords

def tetrads(scale):
    """Generate a list of known four-element chords in a scale."""
    n = len(scale)
    tetrads = {}
    concords = CONCORDS.keys()
    for i in range(n):
        for j in range(i, n):
            for k in range(j, n):
                for l in range(k, n):
                    chord = gamut2identities
                            ([scale[i], scale[j], scale[k], scale[l]])
                    chord.sort()
                    chord = tuple(chord)
                    if chord in concords:
                        tetrads[(scale[i], scale[j], scale[k], scale[l])]
                                = CONCORDS[chord]
    return tetrads
```

Figure A.12: Triads, and tetrads lookup

Stellenbosch University https://scholar.sun.ac.za

```
def tetrachords(scale):
    """Generate a list of known tetrachord elements in a scale."""
    n = len(scale)
    tchords = {}
    tconcords = TCHORDS.keys()
    for i in range(n):
        for j in range(i, n):
            for k in range(j, n):
                for l in range(k, n):
                    tchord = gamut2identities
                                ([scale[i], scale[j], scale[k], scale[l]])
                    tchord.sort()
                    tchord = tuple(tchord)
                    if tchord in tconcords:
                        tchords[(scale[i], scale[j], scale[k], scale[l])]
                                    = TCHORDS[tchord]
    return tchords
```

Figure A.13: Tetrachord lookup

Stellenbosch University https://scholar.sun.ac.za

```python
def scales(scale):
    """Generate a list of known scale elements in a scale."""
    n = len(scale)
    schords = {}
    sconcords = SCAL.keys()
    for i in range(n):
        for j in range(i, n):
            for k in range(j, n):
                for l in range(k, n):
                    for o in range(l, n):
                        for p in range(o, n):
                            for q in range(p, n):
                                schord = gamut2identities
                                            ([scale[i], scale[j], scale[k],
                    scale[l], scale[o], scale[p], scale[q]])
                                schord.sort()
                                schord = tuple(schord)
                                if schord in sconcords:
                                    schords[(scale[i], scale[j], scale[k],
                                            scale[l],scale[o], scale[p], scale[q]
                        = SCAL[schord]
    return schords
```

Figure A.14: Scale lookup

Stellenbosch University https://scholar.sun.ac.za

```
def Ptolemaic12(self):
        """
        ## Simpler PTOLEMAIC TUNING ###
        returns 37 intervals (or 39 incl. extreme commas)
        """
        ptl = []
        PtlX = [Fraction(81, 80)]*13
        PtlX2 = np.prod([T.PY, PtlX], axis=0)
        ptl.append(PtlX2)

        PTL = np.unique(np.concatenate(ptl))
        #print PTL
        PTLO = []
        PToctaves = []
        for k in PTL:
            if Fraction(k)*Fraction(1,2) in PTL:
                PToctaves.append(k)
            if Fraction(k)*Fraction(1,2) not in PTL:
                PTLO.append(k)

        PTLa = []
        ptla = []
        PtlY = [Fraction(80, 81)]*13


        PtlY2 = np.prod([T.PY, PtlY], axis=0)
        ptla.append(PtlY2)

        PTLa = np.unique(np.concatenate(ptla))

        PTLU = []
        PTuctaves = []
        for k in PTLa:
            if Fraction(k)*Fraction(1,2) in PTLa:
                PTuctaves.append(k)
            if Fraction(k)*Fraction(1,2) not in PTLa:
                PTLU.append(k)

        Ptolemaico = np.unique(PTLO + PTLU + T.PY)
        Ptolemaic = Ptolemaico[1:len(Ptolemaico)-1]

        return Ptolemaic
```

Figure A.15: Simple Ptolemaic 5-limit Tuning

Stellenbosch University https://scholar.sun.ac.za

```python
def PtolemaicExt(self):
       #Pythagorean Extended Tuning code goes here, then:
        ptol = []  ## calculate commas acute
        PtolX = [Fraction(81, 80)]*47
        PtolX2 = np.prod([PTO, PtolX], axis=0)
        ptol.append(PtolX2)
        PTOL = np.unique(np.concatenate(ptol))
        PTOLO = []
        PToloctaves = []
        for k in PTOL:
            if Fraction(k)*Fraction(1,2) in PTOL:
                PToloctaves.append(k)
            if Fraction(k)*Fraction(1,2) not in PTOL:
                PTOLO.append(k)
        for p in PTOLO:
            fu = Fraction(p)*Fraction(1,2)
            if fu > Fraction(1,1):
                PTOLO.remove(p)
                PTOLO.append(fu)
        ptola = []        ##calculate comma grave
        PtolY = [Fraction(80, 81)]*47
        PtolY2 = np.prod([PTO, PtolY], axis=0)
        ptola.append(PtolY2)
        PTOLa = np.unique(np.concatenate(ptola))
        PTOLU = []
        PToluctaves = []
        for k in PTOLa:
            if Fraction(k)*Fraction(1,2) in PTOLa:
                PToluctaves.append(k)
            if Fraction(k)*Fraction(1,2) not in PTOLa:
                PTOLU.append(k)
        for p in PTOLU:
            fuu = Fraction(p)*Fraction(1,2)
            if fuu > Fraction(1,1):
                PTOLU.remove(p)
                PTOLU.append(fuu)
        for p in PTOLU:
            fu1 = Fraction(p)*Fraction(1,2)
            if fu1 > Fraction(1,1):
                PTOLU.remove(p)
                PTOLU.append(fu1)
        PtolemaicoX = np.unique(PTOLO + PTOLU + PTO)
        PtolemaicX = PtolemaicoX[1:len(PtolemaicoX)-1]
        return PtolemaicX
```

Figure A.16: Extended Ptolemaic Tuning from Extended Pythagorean Tuning

Stellenbosch University https://scholar.sun.ac.za

```python
def FiveLim(self):
        """Returns 152 5-limit Intervals including the
            24 MT.PYX Intervals and Octave.  """
        pt=[]
        for i in MT.PYX[0:24]:
            PyX = [Fraction(i)]*25
            PyX2 = np.prod([MT.PYX, PyX], axis=0)
            pt.append(PyX2)
        pt.append(MT.PYX)
        PT = np.unique(np.concatenate(pt))
        octaves = []
        PTO = []
        for j in PT:
            if Fraction(j)*Fraction(1,2) in PT:
                octaves.append(j)
            if Fraction(j)*Fraction(1,2) not in PT:
                PTO.append(j)
        for p in PTO:
            d = Fraction(p)*Fraction(1,2)
            if d > Fraction(1,1):
                PTO.remove(p)
                PTO.append(d)
        ptol = []
        PtolX = [Fraction(81, 80)]*47
        PtolX2 = np.prod([PTO, PtolX], axis=0)
        ptol.append(PtolX2)
        PTOL = np.unique(np.concatenate(ptol))
        PTOLO = []
        PToloctaves = []
        for k in PTOL:
            if Fraction(k)*Fraction(1,2) in PTOL:
                PToloctaves.append(k)
            if Fraction(k)*Fraction(1,2) not in PTOL:
                PTOLO.append(k)
        for p in PTOLO:
            fu = Fraction(p)*Fraction(1,2)
            if fu > Fraction(1,1):
                PTOLO.remove(p)
                PTOLO.append(fu)
```

Figure A.17: 5-limit Tuning, pt. 1

Stellenbosch University https://scholar.sun.ac.za

```
ptola = []
PtolY = [Fraction(80, 81)]*47
PtolY2 = np.prod([PTO, PtolY], axis=0)
ptola.append(PtolY2)
PTOLa = np.unique(np.concatenate(ptola))
PTOLU = []
PToluctaves = []
for k in PTOLa:
    if Fraction(k)*Fraction(1,2) in PTOLa:
        PToluctaves.append(k)
    if Fraction(k)*Fraction(1,2) not in PTOLa:
        PTOLU.append(k)
for p in PTOLU:
    fuu = Fraction(p)*Fraction(1,2)
    if fuu > Fraction(1,1):
        PTOLU.remove(p)
        PTOLU.append(fuu)
for p in PTOLU:
    fu1 = Fraction(p)*Fraction(1,2)
    if fu1 > Fraction(1,1):
        PTOLU.remove(p)
        PTOLU.append(fu1)
PtolemaicoX = np.unique(PTOLO + PTOLU + PTO)
PtolemaicX = PtolemaicoX[1:len(PtolemaicoX)-1]
fivelimadds = []
for n in PtolemaicX:
    if n not in MT.PYX:
        fivelimadds.append(n)
fivelim = np.unique(MT.PYX + fivelimadds)
return fivelim
```

Figure A.18: 5-limit Tuning, pt. 2

Stellenbosch University https://scholar.sun.ac.za

```python
def FiveLimAdds(self):

        """
        Extracts 127 5-limit additions from those ratios already
        present in Pythagorean Extended tuning. Returns only additions.
        """
       #same code as 5-limit Tuning but instead:
        return fivelimadds
```

Figure A.19: 5-limit Additions

```python
def CompFiveLim(self):
        """
            Returns Complete 5-limit Intervals including the
            24 MT.PYX Intervals and Octave.
        """
       # 5-limit Tuning code goes here, then:

        c5=[]
        for c in fivelim:
            ca = [Fraction(c)]*len(fivelim)
            caa = np.prod([fivelim, ca], axis=0)
            c5.append(caa)

        Com5lim = np.unique(np.concatenate(c5))

        Comp5lim = [Com5lim[0]]
        for e in Com5lim[1:]:
            fic = Fraction(e)*Fraction(1,2)
            if Fraction(1,1) < fic <= Fraction(2,1):
                Comp5lim.append(fic)
                if fic > Fraction(2,1):
                    fac = fic * Fraction(1,2)
                    if Fraction(1,1) < fac <= Fraction(2,1):
                        Comp5lim.append(fic)

        Comp5lim = np.unique(Comp5lim)
        return Comp5lim
```

Figure A.20: Complete 5-limit Tuning

Stellenbosch University https://scholar.sun.ac.za

```
def ntet(self, num_steps):
        s =self.num_steps
        x = T.generalized_equal_temperament(frame_interval=2, num_steps=s)
        return x
```

Figure A.21: n-TET Decimal Value

Stellenbosch University https://scholar.sun.ac.za

```
def PY440(self):
    return T.to_scale_freqs(T.PY, f0=440)

def PYX440(self):
    return T.to_scale_freqs(MT.PYX, f0=440)

def AUG440(self):
    return T.to_scale_freqs(MT.AUG, f0=440)

def DIM440(self):
    return T.to_scale_freqs(MT.DIM, f0=440)

def EQ440(self):
    return T.to_scale_freqs(T.EQ, f0=440)

def GEQ440(self, num_steps=12):
    return T.to_scale_freqs(T.generalized_equal_temperament(
            num_steps=self.num_steps), f0=440)
```

Figure A.22: Reference Ratios at 440hz

Figure A.23 depict a series of sample reference pitch sets providing various tunings systems as frequencies, from 435Hz to 870Hz, and from 432Hz to 864Hz respectively .

A.1.4 tuningtrans

This module makes use of classes in `librosa` and `ratio` to simplify otherwise lengthy and repetitive code.

Figure A.24 lists the unique pitch frequencies returned from LibRosa's pitch analysis method, counts their occurences, and returns their rounds their values to one decimal place.

Figures A.25, A.26, and A.27 determines a set of best guess tonic frequencies based on a correlation of the number of frequency occurences and their relative magnitudes.

Figure A.28 returns all pitches as their octave equivalents located within an ascending octave from the tonic determined above.

Figure A.29 takes this returned octave gamut and rounds the values to one decimal place.

Figure A.30 tests this octave gamut against the reference pitches of a rational tuning system constructed from the same tonic, and returns lists of both the reference pitches and the matching frequencies, as well as the ratios of both the aforementioned lists.

Stellenbosch University https://scholar.sun.ac.za

```python
def PY435(self):
        """reference frequencies 435Hz"""
        return T.to_scale_freqs(T.PY, f0=435)

def PYX435(self):
        return T.to_scale_freqs(MT.PYX, f0=435)

def AUG435(self):
        return T.to_scale_freqs(MT.AUG, f0=435)

def DIM435(self):
        return T.to_scale_freqs(MT.DIM, f0=435)

def EQ435(self):
        return T.to_scale_freqs(T.EQ, f0=435)

def GEQ435(self, num_steps=12):
        return T.to_scale_freqs(T.generalized_equal_temperament(
                num_steps=self.num_steps), f0=435)

def PY432(self):
        """reference frequencies 432Hz"""
        return T.to_scale_freqs(T.PY, f0=432)

def PYX432(self):
        return T.to_scale_freqs(MT.PYX, f0=432)

def AUG432(self):
        return T.to_scale_freqs(MT.AUG, f0=432)

def DIM432(self):
        return T.to_scale_freqs(MT.DIM, f0=432)

def EQ432(self):
        return T.to_scale_freqs(T.EQ, f0=432)

def GEQ432(self, num_steps=12):
        return T.to_scale_freqs(T.generalized_equal_temperament(
                num_steps=self.num_steps), f0=432)
```

Figure A.23: Alternate Reference Ratios

Stellenbosch University https://scholar.sun.ac.za

```python
def countpitch(self, pitches):
        """
        count pitches returned from librosa
        get values rounded to one decimal place
        eg. c = TT.countpitch(pitches)
        """
        #find unique pitches
        #p = np.unique(pitches) # too exact for matching, round

        #round unique pitches and ouput arrays to single array
        b = pitches.tolist()

        #list all pitches to one decimal place
        c = []
        for x in b:
            for y in x:
                if y > 0:
                    c.append(round(y.real, ndigits=1))
        return c
```

Figure A.24: Counting Pitches

Figure A.31 plots these results.

Figure A.32 returns all the original pitches as octave equivalents located within an ascending octave from an equal tempered pitch set proceeding from A = 440Hz, or any other of this sets equally tempered frequencies, such as D = 293.6Hz.

Figure A.33 takes this returned tempered octave gamut and rounds the values to one decimal place.

Figure A.34 tests this tempered octave gamut against the reference pitches of a temperament scheme constructed from the same tempered tonic, and returns lists of both the reference pitches and the matching frequencies, as well as the tempered scale step-numbers of the frequencies in both the aforementioned lists.

Figure A.35 plots these results.

Figure A.36 returns data arrays of pitches and magnitudes from a wave file.

Figure A.37 also return data arrays of pitches and magnitudes from a wave file but the maximum and minimum frequencies have been limited to the range of the cello used in this sample, reducing the artifacts introduced by inflections, noises and sympathetic resonances .

Figure A.38 shows the plotting of various aspects of the above feature extractions.

Stellenbosch University https://scholar.sun.ac.za

```
def testdomfreq(self, pitches, magnitudes):
        p = np.unique(pitches)
        #m = np.unique(magnitudes)
        b = pitches.tolist()
        w = magnitudes.tolist()
        c = []
        counter = 0
        for x in magnitudes:
            for y in x:
                if y > counter:
                    counter = y
                    c.append(y)
        d = []
        for x in b:
            for y in x:
                d.append(y)
        e = []
        for x in w:
            for y in x:
                e.append(y)
        highest = e.index(c[len(c)-1]) #last index, highest mag pitch index
        nexthighest = e.index(c[len(c)-2]) #second last index
        thirdhighest = e.index(c[len(c)-3]) #and so on
        fourthighest = e.index(c[len(c)-4]) #and so on
        fifthighest = e.index(c[len(c)-5]) #and so on
        sixthighest = e.index(c[len(c)-6]) #and so on
        seventhighest = e.index(c[len(c)-7]) #and so on
        eighthighest = e.index(c[len(c)-8]) #and so on
        highestm = d[highest]
        nexthighestm = d[nexthighest]
        thirdhighestm = d[thirdhighest]
        fourthighestm = d[fourthighest]
        fifthighestm = d[fifthighest]
        sixthighestm = d[sixthighest]
        seventhighestm = d[seventhighest]
        eighthighestm = d[eighthighest]
```

Figure A.25: Testing for Tonic / Generator, pt.1

Stellenbosch University https://scholar.sun.ac.za

```
def testdomfreq(self, pitches, magnitudes): #contd
        numhighestm = []
        numnexthighestm = []
        numthirdhighestm = []
        numfourthighestm = []
        numfifthighestm = []
        numsixthighestm = []
        numseventhighestm = []
        numeighthighestm = []
        tempy = []
        for x in p:
            tempy.append(round(x, ndigits=1))
        for x in tempy:
            if x == round(highestm, ndigits=1):
                numhighestm.append(x)
            if x == round(nexthighestm, ndigits=1):
                numnexthighestm.append(x)
            if x == round(thirdhighestm, ndigits=1):
                numthirdhighestm.append(x)
            if x == round(fourthighestm, ndigits=1):
                numfourthighestm.append(x)
            if x == round(fifthighestm, ndigits=1):
                numfifthighestm.append(x)
            if x == round(sixthighestm, ndigits=1):
                numsixthighestm.append(x)
            if x == round(seventhighestm, ndigits=1):
                numseventhighestm.append(x)
            if x == round(eighthighestm, ndigits=1):
                numeighthighestm.append(x)
```

Figure A.26: Testing for Tonic / Generator, pt. 2

Stellenbosch University https://scholar.sun.ac.za

```
def testdomfreq(self, pitches, magnitudes): #contd
        domf0 = []
        counterf = 0
        if len(numhighestm) > counterf:
            counterf = len(numhighestm)
            domf0 = round(highestm, ndigits=1)
        if len(numnexthighestm) > counterf:
            counterf = len(numnexthighestm)
            domf0 = round(nexthighestm, ndigits=1)
        if len(numthirdhighestm) > counterf:
            counterf = len(numthirdhighestm)
            domf0 = round(thirdhighestm, ndigits=1)
        if len(numfourthighestm) > counterf:
            counterf = len(numfourthighestm)
            domf0 = round(fourthighestm, ndigits=1)
        if len(numfifthighestm) > counterf:
            counterf = len(numfifthighestm)
            domf0 = round(fifthighestm, ndigits=1)
        if len(numsixthighestm) > counterf:
            counterf = len(numsixthighestm)
            domf0 = round(sixthighestm, ndigits=1)
        if len(numseventhighestm) > counterf:
            counterf = len(numseventhighestm)
            domf0 = round(seventhighestm, ndigits=1)
        if len(numeighthighestm) > counterf:
            counterf = len(numeighthighestm)
            domf0 = round(eighthighestm, ndigits=1)
        return domf0
```

Figure A.27: Testing for Tonic / Generator, pt. 3

Stellenbosch University https://scholar.sun.ac.za

```
def domfreqoctave(self, domf0, c):
        """
        convert c (from countpitch) to one octave from dominant frequency
        start with 256 Hz, and replace with best guess dominant frequency
        eg. coct = TT.domfreqoctave(domf0=256, c)
        """
        domfreq = domf0 #Hz number #391 for gmin #326 for 7sec
        coct = []
        for x in c:
            if domfreq <= x < (domfreq*2):
                coct.append(x)
            if x > (domfreq*2):
                y = x/2
                if domfreq <= y < (domfreq*2):
                    coct.append(y)
                if y > (domfreq*2):
                    z = y/2
                    if domfreq <= z < (domfreq*2):
                        coct.append(z)
                    if z > (domfreq*2):
                        zz = z/2
                        if domfreq <= zz < (domfreq*2):
                            coct.append(zz)

            if x < domfreq :
                y = x*2
                if domfreq  <= y < (domfreq*2):
                    coct.append(y)
        return coct
```

Figure A.28: Octave from Tonic / Generator

Stellenbosch University https://scholar.sun.ac.za

```python
def roundcoct(self, coct):
        """
        Round one octave (coct from domfreqoctave) output
        to one decimal place. Count instances and return:

        pitchouty, -----pitch instances
        and pitchoutz, ------no. of occurences

        eg. pitchouty, pitchoutz = roundcoct(coct)
        """
        #remove decimal places and find unique pitches
        pu = []
        for x in coct:    #replace coct with c to find all unique pitches
        #    d = round(x)
        #    pu.append(d)
            pu.append(x)
        pup = np.unique(pu)

        #count number of times unique pitches occur
        pc = []
        for x in pup:
            if x in coct:
                pc.append([x, coct.count(x)])

        ##reformat unique pitches and count
        pitchouty = [] #pitch identities
        pitchoutz = [] #number of times

        for x in pc:
            pitchouty.append(x[0])
            if x[1] > 1:
                pitchoutz.append(x[1])
            if x[1] <= 1:
                pitchoutz.append(0)
        #this last iteration excludes near null values
        #that were rounded up to one
        return pitchouty, pitchoutz
```

Figure A.29: Implementing pitch value rounding to one decimal place

Stellenbosch University https://scholar.sun.ac.za

```
def ratuningtest(self, tuning_system, pitchouty, domf0):
        """
        Take tuning_system from BregmanTK to compare with
        pitchouty against domf0.
            T.PY, T.JI, MT.PYX, MT.AUG, MT.DIM, but not T.EQ
                RM.LIM5, RM.LIM5ADD, RM.LIM5COM
        Returns freqs, the tuning system test frequencies
        and freqlistr, the list of matches in pitchouty.
        eg. freqlistr, freqs, ratlist, rats = TT.ratuningtest(
                T.PY, pitchouty, domf0)
        """
        tuningsystem = tuning_system
        rats = []
        for x in tuningsystem:
            rats.append(ratio.Ratio(x.numerator, x.denominator))

        freqs = T.to_scale_freqs(tuningsystem, f0=domf0)

        fround = []
        for x in freqs:
            fround.append(round(x, ndigits=1))

        FROU = dict(enumerate(fround))
        FREQ = dict(enumerate(freqs))
        RATS = dict(enumerate(rats))

        def find_key(input_dict, value):
            return next((k for k, v in input_dict.items() if v == value),
                None)
        ratlist = []
        freqlistr = []
        for x in pitchouty:
            y = find_key(FROU, x)
            if y >= 0:
                print 'Scale step number %s at %s Hz' %(y, x)
                freqlistr.append(x)

            for key, value in RATS.items():
                if key == y:
                    print 'with a ratio of %s' %value
                    print ''
                    ratlist.append(value)

        return freqlistr, freqs, ratlist, rats
```

Figure A.30: Test against Tuning Systems

Stellenbosch University https://scholar.sun.ac.za

```python
def ratuningtestplot(self, freqlistr, freqs, ratlist):
    """
    Plot Test ratio frequencies against data
    eg. plt.figure(4)
        TT.ratuningtestplot(freqlistr, freqs, ratlist)
        plt.title('Rational Tuning')
        plt.show(4)
    """
    plt.subplot(211)
    plt.plot(list(freqlistr),'ro')
    plt.xlabel('%s Pitches found' %s(len(freqlistr)-1))
    plt.ylabel('Found Frequencies (Hz)')
    #plt.text(0, 1, '%s' %ratlist)
    plt.subplot(212)
    plt.plot(list(freqs),'bo')
    plt.xlabel('%s Pitches' %(len(freqs)-1))
    plt.ylabel('Sample Pitches (Hz)')
```

Figure A.31: Plot results of Tuning testing

Figure A.39 test tuning ratios returned by the tuning test above for triads, tetrads, tetrachords, and scales respectively, and returns lists of the results.

Figure A.40 show some utilities for converting fractional notation to ratio.

Stellenbosch University https://scholar.sun.ac.za

```
def eqfreqoctave(self, c, eqfreq = 440):
        """
        convert c (from countpitch) to one octave from A = 440 Hz
        eg. eqcoct = TT.eqfreqoctave(c)
        """
        eqcoct = []
        for x in c:
            if eqfreq <= x < (eqfreq*2):
                eqcoct.append(x)
            if x > (eqfreq*2):
                y = x/2
                if eqfreq <= y < (eqfreq*2):
                    eqcoct.append(y)
                if y > (eqfreq*2):
                    z = y/2
                    if eqfreq <= z < (eqfreq*2):
                        eqcoct.append(z)
                    if z > (eqfreq*2):
                        zz = z/2
                        if eqfreq <= zz < (eqfreq*2):
                            eqcoct.append(zz)

            if x < eqfreq :
                y = x*2
                if eqfreq  <= y < (eqfreq*2):
                    eqcoct.append(y)
        return eqcoct
```

Figure A.32: Generate Tempered Octave from A=440Hz

Stellenbosch University https://scholar.sun.ac.za

```
def roundeqcoct(self, eqcoct):
        """
        Round one octave (eqcoct from eqfreqoctave) output
        to one decimal place. Count instances and return
        pitchouteqy, -----pitch instances
        and pitchouteqz, ------no. of occurences
        eg. pitchouteqy, pitchouteqz = TT.roundeqcoct(eqcoct)
        """
        #remove decimal places and find unique pitches
        pu = []
        for x in eqcoct:
        #replace coct with c to find all unique pitches
            pu.append(x)
        pup = np.unique(pu)

        #count number of times unique pitches occur
        pc = []
        for x in pup:
            if x in eqcoct:
                pc.append([x, eqcoct.count(x)])

        ##reformat unique pitches and count
        pitchouteqy = [] #pitch identities
        pitchouteqz = [] #number of times

        for x in pc:
            pitchouteqy.append(round(x[0], ndigits=1))
            if x[1] > 1:
                pitchouteqz.append(x[1])
            if x[1] <= 1:
                pitchouteqz.append(0)
        #this last iteration excludes near null values
        #that were rounded up to one
        return pitchouteqy, pitchouteqz
```

Figure A.33: Round Values for Tempered Octave

Stellenbosch University https://scholar.sun.ac.za

```
def eqtuningtest(self, ntet_system, pitchouteqy, domf0):
        """         for nTETs
        ntet_system takes BregmanTK EQ
        and generalized_equal_temperament
        eg. freqlist, notfreqlist, feqround, freqlistless,
                notfreqlistless = TT.eqtuningtest(
        T.EQ, pitchouteqy, domf0)
        returns freqlist, notfreqlist, feqround,
                freqlistless, notfreqlistless"""
        eqfloat = []
        for x in ntet_system:
            eqfloat.append(x)
        eqfreqs = T.to_scale_freqs(ntet_system, f0=domf0)
        feqround = []         #round freqs
        for x in eqfreqs:
            feqround.append(round(x, ndigits=1))
        feqroundless = []
        for x in eqfreqs:
            feqroundless.append(round(x, ndigits=0))
        EQFROU = dict(enumerate(feqround))
        EQFROUL = dict(enumerate(feqroundless))
        def find_key(input_dict, value):
            return next((k for k, v in input_dict.items() if v == value), None)
        notfreqlist = []
        freqlist = []
        for x in pitchouteqy:
            y = find_key(EQFROU, x)
            if y >= 0:
                freqlist.append(x)
            if y == None:
                notfreqlist.append(x)
        notfreqlistless = []
        freqlistless = []
        for x in pitchouteqy:
            z = round(x, ndigits=0)
            y = find_key(EQFROUL, z)
            if y >= 0:
                print 'Scale step number %s at %s Hz' %(
                        y, z) # Frequency in Hz
                freqlistless.append(z)
            if y == None:
                notfreqlistless.append(z)
        return freqlist, notfreqlist, feqround,
                freqlistless, notfreqlistless
```

Figure A.34: Test against Temperament schemes

Stellenbosch University https://scholar.sun.ac.za

```python
def eqtuningtestplot(self, freqlist, eqfreqs):
        """
        Plot Test nTET frequencies against data
        eg. plt.figure(5)
            TT.eqtuningtestplot(freqlist, eqfreqs)
            plt.title('Equal Tuning')
            plt.show(5)
        """
        plt.subplot(211)
        plt.plot(list(freqlist),'ro')
        plt.xlabel('No. of Pitches found')
        plt.ylabel('Found Frequencies(Hz)')
        plt.subplot(212)
        plt.plot(list(eqfreqs),'bo')
        plt.xlabel('No. of Sample Pitches')
        plt.ylabel('%s -TET Sample Pitches (Hz)' %(len(eqfreqs)-1))
```

Figure A.35: Plot results of Temperament testing

```python
def loadwave(self, wav):
        """
        load audio file and get pitches and magnitudes
        using librosa.

        eg. pitches, magnitudes = TT.loadwave('gmin.wav')

        """
        audio_file = wav
        y, sr = librosa.load(audio_file, sr=22050, mono=True, offset=0.0,
        duration=None, dtype=np.float32)
        pitches, magnitudes = librosa.piptrack(y=y, sr=sr, threshold=0.1)

        return pitches, magnitudes
```

Figure A.36: Returning pitches and magnitudes from wave audio

Stellenbosch University https://scholar.sun.ac.za

```python
def loadwavecello(self, wav):
        """
        load audio file and get pitches and magnitudes
        using librosa. Limit to pitches between 64 and 440
        i.e. C2 to A4

        eg. pitches, magnitudes = TT.loadwave('gmin.wav')

        """
        audio_file = wav
        y, sr = librosa.load(audio_file, sr=22050, mono=True,
                offset=0.0, duration=None, dtype=np.float32)
        pitches, magnitudes = librosa.piptrack(y=y, sr=sr,
                fmin=64, fmax=440, threshold=0.1)

        return pitches, magnitudes
```

Figure A.37: Returning only pitches and magnitudes limited to the range of a cello

Stellenbosch University https://scholar.sun.ac.za

```
 def pitchoutyplot(self, pitchouty, pitchoutz, domf0):
    """          Plotting number of Pitch Occurences over Single Octave
    eg. plt.figure(1)
        TT.pitchoutyplot(pitchouty, pitchoutz, domf0)
        plt.title('Pitch Occurences')
        plt.show(1)         """
    plt.xlabel('Frequencies(Hz) transposed to single 8ve')
    plt.ylabel('Number of occurences')
    plt.plot(pitchouty, pitchoutz)
    plt.axis(xmin=domf0, xmax=domf0*2)
def magpitchplot(self, pitches, magnitudes, domf0):
    """          Plotting magnitudes of Pitch Occurences
    eg. plt.figure(2)
        TT.magpitchplot(pitches, magnitudes, domf0)
        plt.title('Pitch Magnitudes')
        plt.show(2)         """
    plt.plot(pitches, magnitudes, 'bo')
    plt.ylabel('Magnitudes')
    plt.xlabel('Frequencies(Hz)')
    plt.axis(xmin=domf0, xmax=domf0*2)
def combiplot(self, pitches, magnitudes, pitchouty, pitchoutz, domf0):
    """          both above plottings as subplots
    eg. plt.figure(3)
        TT.combiplot(pitches, magnitudes, pitchouty, pitchoutz, domf0)
        plt.title('Pitch Magnitudes and Occurences')
        plt.show(3)         """
    plt.subplot(211)
    plt.plot(pitches, magnitudes, 'bo')
    plt.ylabel('Magnitudes')
    plt.xlabel('Frequencies(Hz)')
    plt.axis(xmin=domf0, xmax=domf0*2)
    plt.subplot(212)
    plt.xlabel('Frequencies(Hz) transposed to single 8ve')
    plt.ylabel('Number of occurences')
    plt.plot(pitchouty, pitchoutz)
    plt.axis(xmin=domf0, xmax=domf0*2)
```

Figure A.38: More plotting options

Stellenbosch University https://scholar.sun.ac.za

```
def triads(self, BregTKratios):
    """             Finds known triads from ratios
            (dict in ratio.py)          """
    Triads = ratio.triads(BregTKratios)
    TriadsNum = len(Triads)
    print '%s known triads:' %TriadsNum
    print Triads
    return Triads, TriadsNum
def tetrads(self, BregTKratios):
    """             Finds known tetrads from ratios
            (dict in ratio.py)          """
    Tetrads = ratio.tetrads(BregTKratios)
    TetradsNum = len(Tetrads)
    print '%s known tetrads:' %TetradsNum
    print Tetrads
    return Tetrads, TetradsNum
def tetrachords(self, BregTKratios):
    """             Finds known Tetrachords from ratios
            (dict in ratio.py)          """
    Tetrachords = ratio.tetrachords(BregTKratios)
    TetrachordsNum = len(Tetrachords)
    print '%s known tetrachords:' %TetrachordsNum
    print Tetrachords
    return Tetrachords, TetrachordsNum
def scales(self, BregTKratios):
    """             Finds known scales from ratios
            (dict in ratio.py)          """
    Scales = ratio.scales(BregTKratios)
    ScalesNum = len(Scales)
    print '%s known scales:' %ScalesNum
    print Scales
    return Scales, ScalesNum
```

Figure A.39: Test for Triads, Tetrads, Tetrachords, Scales and logging output

Stellenbosch University https://scholar.sun.ac.za

```
def frac2ratio(self, fractions):
        """             Convert BregmanTK Fractions to Ratios          """
        BregTKratios = []
        scaletype = fractions
        for x in scaletype:
            y = ratio.Ratio(x.numerator, x.denominator)
            BregTKratios.append(y)
        return BregTKratios

def ratiotest(self, ratlist):
        """             Convert BregmanTK Fractions to Ratios          """
        BregTKratios = []
        scaletype = ratlist
        for x in scaletype:
            y = ratio.Ratio(x.num, x.denom)
            BregTKratios.append(y)
        print '%s Ratios from BregmanTK:' %len(BregTKratios)
        print BregTKratios
        return BregTKratios
```

Figure A.40: Fraction to ratio conversion utilities

Stellenbosch University https://scholar.sun.ac.za

Appendix B

Appendix B: Sample Analysis

B.1 Python Analysis Example

B.1.1 Methodology and Results

Two cellists, one a student and the other a professional, were asked to perform a small piece of music, upon the sample cello and with the same bow. The first four bars of J. S. Bach's Cello Suite No. 2 (BWV 1008), up until the point indicated in figure B.1 below, were interpreted by each of them in this sample analysis. They each performed a number of takes, until they were themselves satisfied at their performance. Their performances were recorded using an AKG C414 EB condenser microphone[1] and a Zoom H4 field recorder[2], in Studio A at the University of Stellenbosch Conservatory.

A further MIDI performance was rendered from a midi file, transcribed by David J. Grossman, sequenced and recorded using a sine wave synthesizer in the Ableton Live DAW.

Three audio files, one each from the student and professional cellists, together with the MIDI performance were edited into individual clips using Ableton Live[3].

[1] Set to cardoid, and placed directly in front of the cello at a distance of 60cm from the center of the string length, and a height of 94 cm from the ground.

[2] Set to record 24bit mono wave files at 44.1 kHz.

[3] These were also mixed down to 44.1 kHz 16bit mono wave files for compatibility with the analysis methods developed. 24bit data chunks being incompatible with the float handling

Figure B.1: The first four bars of J. S. Bach's Cello Suite No. 2 (BWV 1008)

Stellenbosch University https://scholar.sun.ac.za

These clips were analysed by the python script below in section B.1.2 using the modules and definitions described in appendix A. Their respective analyses are output as text, logged in below in section B.1.3, and graphed data, see figures B.3, B.4, and B.5.

The logged data shows first the fundamental pitch, or tonic, decided upon for each sample using the analysis of a single note clipped from each of the samples. In each case, this was the first sounded note, the tonic of the scored piece of music (each a tone in the region of D3). The log also returns a value relating the dominant frequency across the entire sample. It should be noted how these value do not always coincide.

The text log returns a gamut analysis showing the correlations of the pitches returned from each sample to the respective ratios of a tuning system chosen for cross-analysis[4]. A similar analysis returns the scale step numbers of a temperament chosen for analysis[5].

In a test for melodic and harmonic analysis, pitches within a certain pitch range[6] and above a certain threshold of magnitude are plotted against time and their frequencies analysed against multi-octave reference sets of tuning systems and temperaments. These results are returned as data arrays of found ratios, equal-tempered step numbers, and their respective frequencies, as well as similar list of the reference sets used for matching.

In both stages, gamut analysis and melodic / harmonic analysis, the ratios found are submitted for analysis determining the presence of mathematically defined rational tuned triads, tetrads, tetrachords and scales. These are output in the text log and some interesting results are further presented below

By analysing the log certain facts are immediately apparent. In both the student and the professional cellists samples we see meaningful rational intervals, which are not visible in the midi sample. These are evidenced by the ratios found in their respective gamut analyses and their underlying structures. These are summarised in table B.1.

In the student sample there were seven interval ratios returned by gamut analysis of rational intervals: $^1/_1$, $^9/_8$, $^{32}/_{27}$, $^{19683}/_{16384}$, $^{8192}/_{6561}$, $^4/_3$, $and^{1048576}/_{531441}$.

These were matched to a single dictionary defined tetrachord structure rooted upon the tonic, a Pythagorean minor tetrachord of $^1/_1$, $^9/_8$, $^{32}/_{27}$, and

methods currently implemented by these analysis scripts.

4 In this case, the extended Pythagorean tuning, shown in figure A.2.

5 24-TET. The tuning systems and temperament schemes available for analysis are described further in section A.1. The tuning and temperament schemes used here were chosen as being representative of the complex approach described here, while not being entirely too complex. Similar analysis run against more finely graduated sets of tuning, such as the 5-limit tunings and higher prime numbered n-TET schemes available, provides more complex analysis.

6 In this case, those frequencies between 64Hz and 440Hz, roughly the range from the unstopped pitch of the low cello string (C2) to the stopped octave upon the highest string (A4).

Stellenbosch University https://scholar.sun.ac.za

Sample	Student	Professional	Midi
No. of Ratios	7	14	13
No. of Triads	0	10	0
No. of Tetrads	0	6	0
No. of Tetrachords	1	1	0

Table B.1: Comparison of Rational Interval Tests

$^{4}/_{3}$.

The professional sample furnished fourteen ratios from gamut analysis: $^{1}/_{1}$, $^{256}/_{243}$, $^{65536}/_{59049}$, $^{9}/_{8}$, $^{32}/_{27}$, $^{19683}/_{16384}$, $^{4}/_{3}$, $^{177147}/_{131072}$, $^{1024}/_{729}$, $^{729}/_{512}$, $^{3}/_{2}$, $^{128}/_{81}$, $^{6561}/_{4069}$, $^{243}/_{128}$, and $^{1048576}/_{531441}$.

These ratios were matched to ten known triads. Three Pythagorean major triads were found: one rooted upon the Pythagorean semitone[7], $^{256}/_{243}$, $^{4}/_{3}$, and $^{128}/_{81}$; one rooted upon the *epogdoic* tone, $^{9}/_{8}$, $^{3}/_{2}$, and $^{243}/_{128}$; and one rooted upon the Pythagorean seventh, $^{243}/_{128}$, $^{19683}/_{16384}$, and $^{729}/_{512}$. Three known Pythagorean minor triads were found: one rooted upon the tonic $^{1}/_{1}$, $^{32}/_{27}$, and $^{3}/_{2}$; one rooted upon the Pythagorean seventh, $^{243}/_{128}$, $^{9}/_{8}$, and $^{729}/_{512}$; and one rooted upon an interval a tone($^{9}/_{8}$) above the greater Pythagorean tritone($^{729}/_{512}$), $^{6561}/_{4096}$, $^{243}/_{128}$, and $^{19683}/_{16384}$. Three Pythagorean diminished triads were found: one rooted on the interval a tone above the greater Pythagorean tritone, $^{6561}/_{4096}$, $^{243}/_{128}$, and $^{9}/_{8}$; one rooted on the interval $^{177147}/_{131072}$ [8], $^{6561}/_{4096}$, and $^{243}/_{128}$; one rooted one the epogdoic tone $^{9}/_{8}$, $^{4}/_{3}$, and $^{128}/_{81}$; and one rooted on the Pythagorean seventh $^{128}/_{81}$, $^{9}/_{8}$, and $^{4}/_{3}$.

These ratios were further matched to 6 known tetrads. Three Pythagorean minor sixth tetrads were found: one rooted upon the fourth $^{4}/_{3}$, $^{128}/_{81}$, $^{1}/_{1}$, and $^{9}/_{8}$; one rooted upon the Pythagorean seventh $^{243}/_{128}$, $^{19683}/_{16384}$, $^{729}/_{512}$, and $^{6561}/_{4096}$; and one rooted upon the interval a tone above the greater Pythagorean tritone, $^{6561}/_{4096}$, $^{243}/_{128}$, $^{19683}/_{19384}$, and $^{177147}/_{131072}$. Two Pythagorean seventh tetrads were found: one rooted on the fifth $^{3}/_{2}$, $^{243}/_{128}$, $^{9}/_{8}$, and $^{729}/_{512}$; and another rooted on the Pythagorean semitone $^{256}/_{243}$, $^{4}/_{3}$, $^{128}/_{81}$, and $^{1}/_{1}$. One Pythagorean major sixth tetrad was found, rooted on the Pythagorean seventh $^{243}/_{128}$, $^{19683}/_{16384}$, $^{729}/_{512}$, and $^{6561}/_{4096}$.

One known tetrachord was matched to these ratios, a Pythagorean minor tetrachord rooted upon the interval $^{19683}/_{16384}$ [9], and proceeding $^{177147}/_{131072}$, $^{729}/_{512}$, and $^{6561}/_{4096}$.

Two ratios were returned by the Midi sample, $^{1}/_{1}$, and $^{32}/_{27}$, a single Pythagorean third.

---

[7] A *limma*.

[8] This interval is located a Pythagorean minor third below the interval $^{6561}/_{4096}$.

[9] This interval is a tone($^{9}/_{8}$) below the interval $^{177147}/_{131072}$.

Stellenbosch University https://scholar.sun.ac.za

Figure B.2: Gamut of the first four bars of J. S. Bach's Cello Suite No. 2 (BWV 1008)

These results are interesting for the implications they may provide for schools of pedagogy and taste. The presence of the single tonicrooted minor tetrachord in the student sample and the favouring of harmonic intonation in the professional sample offer interesting comparisons.

Of the notes expected to be found in the gamut of the scored work, we would expect to see the presence of pitches fitting the 12-TET approximations of C♯3, D3, E3, F3, G3, A3, B♭3, C♯4, and E4, the gamut shown in figure B.2. These identities are answered as shown in table B.2, condensed from the output log in section B.1.3.

In addition to these ratios, this sample analysis returned a number of matches to a 24-tet scheme, shown in table B.3.

In addition to these results a series of isolated tonic reference pitches were recorded by the professional cellist for registration of the tonic pitch. These were the open D on the cello's second string, the stopped D on the cello's third string, and this same stopped D with vibrato. The respective differences resulting from their analysis are shown in table B.4. Such pitch variations as these critically influence the analysis this research proposes. This clearly illustrates the need for the introduction of a threshold function for grouping closely related enharmonic intervals for analysis.

Stellenbosch University https://scholar.sun.ac.za

3-lim Interval	Sample 1	Sample 2	Sample 3	12-TET Pitch Class
1/1	146.5 Hz	147.4 Hz	146.5 Hz	D
256/243		155.3 Hz		E♭
65536/59049		163.6 Hz		D♯
9/8	164.8 Hz	165.8 Hz	164.8 Hz	E
32/27	173.6 Hz			F
19683/16384	176.0 Hz	177.1 Hz		F
8192/6561	182.9 Hz			F♯
4/3	195.3 Hz	196.5 Hz		G
177147/131072		199.2 Hz		Fx
1024/729		207 Hz		A♭
729/512		209.9 Hz		G♯
3/2		221.1 Hz		A
128/81		232.9 Hz		B♭
6561/4096		236.1 Hz		B♭
243/128		279.8 Hz		C♯
1048576/531441	289.1 Hz			C♯

Table B.2: Side by side comparison of sample tuning analysis results

24-tet Step No.	Sample 1(Hz)	Sample 2	Sample 3	12-tet Pitch Class
0	146.8 Hz	146.8 Hz		D
2				E♭
3			160.1 Hz	E♭
4	164.8 Hz	164.8 Hz	164.8Hz	E
6	174.6 Hz	174.6 Hz		F
7				F
8	185 Hz			F♯
10	196 Hz	196 Hz	196 Hz	G
11	201.7 Hz	201.7 Hz		G
14			220 Hz	A
16	233 Hz		233 Hz	B♭
21		269.2		C♯
22	277.1 Hz		277.1 Hz	C♯

Table B.3: Side by side comparison of sample temperament analysis results

Stellenbosch University https://scholar.sun.ac.za

Student	Professional	Open D string	Stopped D	Stopped D with Vibrato
146.5 Hz	147.4 Hz	147.5 Hz	148 Hz	147.4

Table B.4: Comparison of various tonics

Stellenbosch University https://scholar.sun.ac.za

B.1.2 Analysis Script

The following script returns the logged data presented in section B.1.3 and the graphs in section B.1.4.

```
import librosa
import numpy as np
import matplotlib.pyplot as plt
from bregman import tuning
import moretuning
import ratio
import ratioman
import tuningtranslate

TT = tuningtranslate.TuningTranslating()
RM = ratioman.RatioMan()
T = tuning.TuningSystem()
MT = moretuning.MoreTunings()

##############################
##### SAMPLE 1 - Student #####
##############################

######## load sample reference pitch
print 'SAMPLE 1 - gamut analysis'
y1, sr1, pitches0, magnitudes0 = TT.loadwavest(
    'Example D minor/16bit/Fundamental Student Take 3.wav')
c1 = TT.countpitch(pitches0)
domf1, highestm, nexthighestm, thirdhighestm, fourthighestm, fifthighestm,
    sixthighestm, seventhighestm, eighthighestm = TT.retestdomfreq(
    pitches0, magnitudes0, c1)
print 'fundamental: %s' %domf1

###Load Sample
pitches, magnitudes = TT.loadwavecello(
    'Example D minor/16bit/Student Take 3.wav')

####count number of pitch occurences
c = TT.countpitch(pitches)

###find dominant frequency
checkdomf0 = TT.testdomfreq(pitches, magnitudes)
print 'dominant: %s' %checkdomf0
```

Stellenbosch University https://scholar.sun.ac.za

```
###convert count to one octave from domf0 to octave
coct = TT.domfreqoctave(domf1, c)

###take above result and extract instances (y) and no. of occurences (z)
pitchouty, pitchoutz = TT.roundcoct(coct)

#####Test for rational intervals
print 'Rational Intervals Found:'
freqlistr, freqs, ratlist, rats = TT.ratuningtest(
    MT.PYX, pitchouty, domf1)

####Test for triads, tetrads, tetrachords, & scales
BregTKratiosA = np.unique(TT.ratiotest(ratlist))
BTKRA = np.unique(BregTKratiosA)
TriadsA = TT.triads(BTKRA)
TetradsA = TT.tetrads(BTKRA)
TetrachordsA = TT.tetrachords(BTKRA)
ScalesA = TT.scales(BTKRA)

#####EQ tempered Tests
print 'Tempered Intervals Found:'
#Round to one octave from A = 440
eqcoct = TT.eqfreqoctave(c, eqfreq = 146.8)

#return pitch instances (y) and occurences (z)
pitchouteqy, pitchouteqz = TT.roundeqcoct(eqcoct)
pitchouteqyFix = np.unique(pitchouteqy)

#####Test for equal tempered intervals
freqlist, notfreqlist, feqround, freqlistless,
    notfreqlistless = TT.eqtuningtest(
    (T.generalized_equal_temperament(frame_interval=2,
    num_steps=24)), pitchouteqyFix, domf0 = 146.8)

###########################
####  HARMONICMELODIC  ####
###########################
print 'SAMPLE 1- melodic analysis'
xtrm, xtr, rea = TT.domfreqovertime(pitches, magnitudes)
melodicharmonic = TT.melodicharmonic(xtrm)

## TEST FOR JI PITCHES
print 'Rational Intervals Found:'
t_system = MT.PYX
```

Stellenbosch University https://scholar.sun.ac.za

```
octavelist = TT.threeoctaves(t_system)
testp = TT.testpitchesovertime(melodicharmonic, domf1)
freqlistr3, freqs3, ratlist3, rats3 = TT.ratuningtest(octavelist,
    pitchouty=testp, domf0=domf1)

req = []
for x in rea:
    for y in x:
        req.append(y)

BregTKratios = np.unique(TT.ratiotest(ratlist3))
BTKR = np.unique(BregTKratios)
Triads = TT.triads(BTKR)
Tetrads = TT.tetrads(BTKR)
Tetrachords = TT.tetrachords(BTKR)
Scales = TT.scales(BTKR)

## TEST FOR EQ TEMPERED PITCHES
print 'Tempered Intervals Found:'


eq_tsystem = RM.ntet24()
#one octave
eqcoct3 = TT.eqfreqoctave(c=req, eqfreq = 146.8)
pitchouteqy3, pitchouteqz3 = TT.roundeqcoct(eqcoct=eqcoct3)
freqlist3, notfreqlist3, feqround3, freqlistless3,
    notfreqlistless3 = TT.eqtuningtest(eq_tsystem,
    pitchouteqy=np.unique(pitchouteqy3), domf0=146.8)
#second octave
eqcoct31 = TT.eqfreqoctave(c=req, eqfreq = 293.6)
pitchouteqy31, pitchouteqz31 = TT.roundeqcoct(eqcoct=eqcoct31)

freqlist31, notfreqlist31, feqround31, freqlistless31,
    notfreqlistless31 = TT.eqtuningtest(eq_tsystem,
    pitchouteqy=np.unique(pitchouteqy31), domf0=293.6)

########## PLOT SAMPLE 1 ############

plt.figure(figsize=(12, 8))

###plotting pitch and magnitudes

plt.subplot(4, 2, 1)
plt.plot(pitches, magnitudes, 'bo')
```

Stellenbosch University https://scholar.sun.ac.za

```
plt.ylabel('Magnitudes')
plt.xlabel('Frequencies(Hz)')
plt.title('Sample 1')

###  plotting pitch occurences
plt.subplot(4, 2, 2)
plt.xlabel('Frequencies(Hz) transposed to single 8ve')
plt.ylabel('No. of occurences')
plt.plot(pitchouty, pitchoutz)
plt.title('Gamut')
plt.axis(xmin=domf1, xmax=domf1*2)

#PLOT JI INTERVALS
plt.subplot(4, 2, 3)
plt.plot(list(freqlistr),'ro')
plt.plot(list(freqs),'bs')
plt.xlabel('%s Sample Pitches, %s Gamut Pitches found' %(
    (len(freqs)-1),(len(freqlistr))))
plt.ylabel('Frequencies (Hz)')
plt.title('Rational Tuning Intervals')

#PLOT EQ INTERVALS
plt.subplot(4, 2, 4)
plt.plot(list(np.unique(freqlist)),'ro')
plt.plot(list(feqround),'bs')
plt.xlabel('%s - TET Sample Pitches, %s Gamut Pitches found' %(
    (len(feqround)-1),(len(np.unique(freqlist)))))
plt.ylabel('Frequencies (Hz)')
plt.title('Equal Tempered Intervals')

###  plotting melodicharmonic pitch occurences
plt.subplot(4, 2, 5)
plt.xlabel('Samples')
plt.ylabel('Pitches')
plt.plot(melodicharmonic,'bo')
plt.title('Melodic Successions & Harmonies')
plt.tight_layout()

#PLOT JI INTERVALS
plt.subplot(4, 2, 6)
plt.plot(list(freqlistr3),'ro')
plt.plot(list(freqs3),'bs')
plt.xlabel('%s Sample Pitches, %s Pitches found' %(
    (len(freqs3)-1),(len(freqlistr3))))
```

Stellenbosch University https://scholar.sun.ac.za

```
plt.ylabel('Found (Hz)')
plt.title('Rational Tuning Intervals')
plt.tight_layout()

#PLOT EQ INTERVALS 1st octave
plt.subplot(4, 2, 7)
plt.plot(list(np.unique(freqlistless3)),'r^')
plt.plot(list(feqround3),'bo')
plt.xlabel('%s - TET Sample Pitches, %s Pitches found' %(
    (len(feqround3)-1),(len(np.unique(freqlistless3)))))
plt.ylabel('Frequencies (Hz)')
plt.title('Equally Tempered Intervals - 1st octave')
plt.tight_layout()

#PLOT EQ INTERVALS 2nd octave
plt.subplot(4, 2, 8)
plt.plot(list(np.unique(freqlistless31)),'r^')
plt.plot(list(feqround31),'bo')
plt.xlabel('%s - TET Sample Pitches, %s Pitches found' %(
    (len(feqround31)-1),(len(np.unique(freqlistless31)))))
plt.ylabel('Frequencies (Hz)')
plt.title('Equally Tempered Intervals - 2nd octave')
plt.tight_layout()

plt.show(1)

###################################
#####      SAMPLE 2 - Professional       #####
###################################
print 'SAMPLE 2 - gamut analysis'
######## LOAD SAMPLE reference pitch
y2, sr2, pitches1, magnitudes1 = TT.loadwavest(
    'Example D minor/16bit/Fundamental Professional Take 5.wav')
c2 = TT.countpitch(pitches1)
domf0, highestm, nexthighestm, thirdhighestm, fourthighestm, fifthighestm,
    sixthighestm, seventhighestm, eighthighestm = TT.retestdomfreq(
    pitches1, magnitudes1, c2)
print 'fundamental: %s' %domf0
##########Load Sample
pitches2, magnitudes2 = TT.loadwavecello(
    'Example D minor/16bit/Professional Take 5.wav')

####count number of pitch occurences
c0 = TT.countpitch(pitches2)
```

Stellenbosch University https://scholar.sun.ac.za

```
###find dominant frequency
checkdomf00 = TT.testdomfreq(pitches2, magnitudes2)
print 'dominant: %s' %checkdomf00

###convert count to one octave from domf0 to octave
coct0 = TT.domfreqoctave(domf0, c0)

###take above result and extract instances (y) and no. of occurences (z)
pitchouty0, pitchoutz0 = TT.roundcoct(coct0)

#####Test for rational intervals
print 'Rational Intervals Found:'
freqlistr0, freqs0, ratlist0, rats0 = TT.ratuningtest(
    MT.PYX, pitchouty0, domf0)

####Test for triads, tetrads, tetrachords, & scales
BregTKratiosB = np.unique(TT.ratiotest(ratlist0))
BTKRB = np.unique(BregTKratiosB)
TriadsB = TT.triads(BTKRB)
TetradsB = TT.tetrads(BTKRB)
TetrachordsB = TT.tetrachords(BTKRB)
ScalesB = TT.scales(BTKRB)

#####EQ tempered Tests
print 'Tempered Intervals Found:'
#Round to one octave from A = 440
eqcoct0 = TT.eqfreqoctave(c0, eqfreq = 146.8)

#return pitch instances (y) and occurences (z)
pitchouteqy0, pitchouteqz0 = TT.roundeqcoct(eqcoct0)
pitchouteqy0Fix = np.unique(pitchouteqy0)

#####Test for equal tempered intervals
freqlist0, notfreqlist0, feqround0, freqlistless0,
    notfreqlistless0 = TT.eqtuningtest(
    (T.generalized_equal_temperament(frame_interval=2,
    num_steps=24)), pitchouteqy0Fix, domf0 = 146.8)

###########################
####  HARMONICMELODIC  ####
###########################
print 'SAMPLE 2 - melodic analysis'
xtrm2, xtr2, rea2 = TT.domfreqovertime(pitches2, magnitudes2)
```

Stellenbosch University https://scholar.sun.ac.za

```
melodicharmonic2 = TT.melodicharmonic(xtrm2)

## TEST FOR JI PITCHES
print 'Rational Intervals Found:'
t_system2 = MT.PYX
octavelist2 = TT.threeoctaves(t_system2)
testp2 = TT.testpitchesovertime(melodicharmonic2, domf0)
freqlistr4, freqs4, ratlist4, rats4 = TT.ratuningtest(octavelist2,
    pitchouty=testp2, domf0=domf0)

req2 = []
for x in rea2:
    for y in x:
        req2.append(y)

BregTKratios2 = np.unique(TT.ratiotest(ratlist4))
BTKR2 = np.unique(BregTKratios2)
Triads2 = TT.triads(BTKR2)
Tetrads2 = TT.tetrads(BTKR2)
Tetrachords2 = TT.tetrachords(BTKR2)
Scales2 = TT.scales(BTKR2)

## TEST FOR EQ TEMPERED PITCHES
print 'Tempered Intervals Found:'
eq_tsystem2 = RM.ntet24()
#first octave
eqcoct4 = TT.eqfreqoctave(c=req2, eqfreq = 146.8)
pitchouteqy4, pitchouteqz4 = TT.roundeqcoct(eqcoct=eqcoct4)
freqlist4, notfreqlist4, feqround4, freqlistless4,
    notfreqlistless4 = TT.eqtuningtest(eq_tsystem,
    pitchouteqy=np.unique(pitchouteqy4), domf0=146.8)
#second octave
eqcoct41 = TT.eqfreqoctave(c=req2, eqfreq = 293.6)
pitchouteqy41, pitchouteqz41 = TT.roundeqcoct(eqcoct=eqcoct41)
freqlist41, notfreqlist41, feqround41, freqlistless41,
    notfreqlistless41 = TT.eqtuningtest(eq_tsystem,
    pitchouteqy=np.unique(pitchouteqy41), domf0=293.6)

########## PLOT SAMPLE 2 ############

plt.figure(figsize=(12, 8))

###plotting pitch and magnitudes
```

Stellenbosch University https://scholar.sun.ac.za

```
plt.subplot(4, 2, 1)
plt.plot(pitches2, magnitudes2, 'bo')
plt.ylabel('Magnitudes')
plt.xlabel('Frequencies(Hz)')
plt.title('Sample 2')

###  plotting pitch occurences
plt.subplot(4, 2, 2)
plt.xlabel('Frequencies(Hz) transposed to single 8ve')
plt.ylabel('No. of occurences')
plt.plot(pitchouty0, pitchoutz0)
plt.title('Gamut')
plt.axis(xmin=domf0, xmax=domf0*2)

#PLOT JI INTERVALS
plt.subplot(4, 2, 3)
plt.plot(list(freqlistr0),'ro')
plt.plot(list(freqs0),'bs')
plt.xlabel('%s Sample Pitches, %s Gamut Pitches found' %(
    (len(freqs0)-1),(len(freqlistr0))))
plt.ylabel('Frequencies (Hz)')
plt.title('Rational Tuning Intervals')

#PLOT EQ INTERVALS
plt.subplot(4, 2, 4)
plt.plot(list(np.unique(freqlist0)),'ro')
plt.plot(list(feqround0),'bs')
plt.xlabel('%s - TET Sample Pitches, %s Gamut Pitches found'  %(
    (len(feqround0)-1),(len(np.unique(freqlist0)))))
plt.ylabel('Frequencies (Hz)')
plt.title('Equal Tempered Intervals')

###  plotting melodicharmonic pitch occurences
plt.subplot(4, 2, 5)
plt.xlabel('Samples')
plt.ylabel('Pitches')
plt.plot(melodicharmonic2,'bo')
plt.title('Melodic Successions & Harmonies')
plt.tight_layout()

#PLOT JI INTERVALS
plt.subplot(4, 2, 6)
plt.plot(list(freqlistr4),'ro')
plt.plot(list(freqs4),'bs')
```

Stellenbosch University https://scholar.sun.ac.za

```
plt.xlabel('%s Sample Pitches, %s Pitches found' %(
    (len(freqs4)-1),(len(freqlistr4))))
plt.ylabel('Found (Hz)')
plt.title('Rational Tuning Intervals')
plt.tight_layout()

#PLOT EQ INTERVALS - 1st octave
plt.subplot(4, 2, 7)
plt.plot(list(np.unique(freqlistless4)),'r^')
plt.plot(list(feqround4),'bo')
plt.xlabel('%s - TET Sample Pitches, %s Pitches found' %(
    (len(feqround4)-1),(len(np.unique(freqlistless4)))))
plt.ylabel('Frequencies (Hz)')
plt.title('Equally Tempered Intervals - 1st octave')
plt.tight_layout()

#PLOT EQ INTERVALS - 2nd octave
plt.subplot(4, 2, 8)
plt.plot(list(np.unique(freqlistless41)),'r^')
plt.plot(list(feqround41),'bo')
plt.xlabel('%s - TET Sample Pitches, %s Pitches found' %(
    (len(feqround41)-1),(len(np.unique(freqlistless41)))))
plt.ylabel('Frequencies (Hz)')
plt.title('Equally Tempered Intervals - 2nd octave')
plt.tight_layout()

plt.show(2)

#################################
#########   SAMPLE 3 MIDI   #########
#################################
print 'SAMPLE 3 - gamut analysis'
######## LOAD SAMPLE reference pitch
ym, srm, pitchesm0, magnitudesm0 = TT.loadwavest(
    'Example D minor/16bit/D minor Bach Cello MIDI.wav')
cm = TT.countpitch(pitchesm0)
domfm, highestm, nexthighestm, thirdhighestm, fourthighestm,
    fifthighestm, sixthighestm, seventhighestm,
    eighthighestm = TT.retestdomfreq(pitchesm0,
    magnitudesm0, cm)
print 'fundamental: %s' %domfm

##########Load Sample
pitchesm, magnitudesm = TT.loadwavecello(
```

Stellenbosch University https://scholar.sun.ac.za

```
    'Example D minor/16bit/D minor Bach Cello MIDI.wav')

####count number of pitch occurences
cm = TT.countpitch(pitchesm)


###find dominant frequency
checkdomfm = TT.testdomfreq(pitchesm, magnitudesm)
print 'dominant: %s' %checkdomfm

###convert count to one octave from domf0 to octave
coctm = TT.domfreqoctave(domfm, cm)


###take above result and extract instances (y) and no. of occurences (z)
pitchoutym, pitchoutzm = TT.roundcoct(coctm)

#####Test for rational intervals
print 'Rational Intervals Found:'
freqlistrm, freqsm, ratlistm, ratsm  = TT.ratuningtest(
    MT.PYX, pitchoutym, domfm)

####Test for triads, tetrads, tetrachords, & scales
BregTKratiosC = np.unique(TT.ratiotest(ratlistm))
BTKRC = np.unique(BregTKratiosC)
TriadsC = TT.triads(BTKRC)
TetradsC = TT.tetrads(BTKRC)
TetrachordsC = TT.tetrachords(BTKRC)
ScalesC = TT.scales(BTKRC)

#####EQ tempered Tests
#Round to one octave from A = 440
print 'Tempered Intervals Found:'
eqcoctm = TT.eqfreqoctave(cm, eqfreq = 146.8)

#return pitch instances (y) and occurences (z)
pitchouteqym, pitchouteqzm = TT.roundeqcoct(eqcoctm)
pitchouteqymFix = np.unique(pitchouteqym)

#####Test for equal tempered intervals
freqlistm, notfreqlistm, feqroundm, freqlistlessm,
    notfreqlistlessm = TT.eqtuningtest(
    (T.generalized_equal_temperament(frame_interval=2,
    num_steps=24)), pitchouteqymFix, domf0 = 146.8)
```

Stellenbosch University https://scholar.sun.ac.za

```
###########################
####  HARMONICMELODIC  ####
###########################
print 'SAMPLE 3 - melodic analysis'
xtrm3, xtr3, rea3 = TT.domfreqovertime(pitchesm, magnitudesm)
melodicharmonic3 = TT.melodicharmonic(xtrm3)

## TEST FOR JI PITCHES
print 'Rational Intervals Found:'
t_system3 = MT.PYX
octavelist3 = TT.threeoctaves(t_system3)
testp3 = TT.testpitchesovertime(melodicharmonic3, domfm)
freqlistr5, freqs5, ratlist5, rats5 = TT.ratuningtest(
    octavelist3, pitchouty=testp3, domf0=domfm)

req3 = []
for x in rea3:
    for y in x:
        req3.append(y)

BregTKratios3 = np.unique(TT.ratiotest(ratlist5))
BTKR3 = np.unique(BregTKratios3)
Triads3 = TT.triads(BTKR3)
Tetrads3 = TT.tetrads(BTKR3)
Tetrachords3 = TT.tetrachords(BTKR3)
Scales3 = TT.scales(BTKR3)

## TEST FOR EQ TEMPERED PITCHES
print 'Tempered Intervals Found:'
eq_tsystem3 = RM.ntet24()
#1st octave
eqcoct5 = TT.eqfreqoctave(c=req3, eqfreq = 146.8)
pitchouteqy5, pitchouteqz5 = TT.roundeqcoct(eqcoct=eqcoct5)
freqlist5, notfreqlist5, feqround5, freqlistless5,
    notfreqlistless5 = TT.eqtuningtest(eq_tsystem,
    pitchouteqy=np.unique(pitchouteqy5), domf0=146.8)
#2nd octave
eqcoct51 = TT.eqfreqoctave(c=req3, eqfreq = 293.6)
pitchouteqy51, pitchouteqz51 = TT.roundeqcoct(eqcoct=eqcoct51)
freqlist51, notfreqlist51, feqround51, freqlistless51,
    notfreqlistless51 = TT.eqtuningtest(eq_tsystem,
    pitchouteqy=np.unique(pitchouteqy51), domf0=293.6)

########## PLOT SAMPLE 3 ############
```

Stellenbosch University https://scholar.sun.ac.za

```
plt.figure(figsize=(12, 8))

plt.subplot(4, 2, 1)
plt.plot(pitchesm, magnitudesm, 'bo')
plt.ylabel('Magnitudes')
plt.xlabel('Frequencies(Hz)')
plt.title('Sample 3')

###  plotting pitch occurences
plt.subplot(4, 2, 2)
plt.xlabel('Frequencies(Hz) transposed to single 8ve')
plt.ylabel('No. of occurences')
plt.plot(pitchoutym, pitchoutzm)
plt.title('Gamut')
plt.axis(xmin=domf0, xmax=domf0*2)

#PLOT JI INTERVALS
plt.subplot(4, 2, 3)
plt.plot(list(freqlistrm),'ro')
plt.plot(list(freqsm),'bs')
plt.xlabel('%s Sample Pitches, %s Gamut Pitches found' %(
    (len(freqsm)-1),(len(freqlistrm))))
plt.ylabel('Frequencies (Hz)')
plt.title('Rational Tuning Intervals')

#PLOT EQ INTERVALS
plt.subplot(4, 2, 4)
plt.plot(list(np.unique(freqlistm)),'ro')
plt.plot(list(feqroundm),'bs')
plt.xlabel('%s - TET Sample Pitches, %s Gamut Pitches found'  %(
    (len(feqround5)-1),(len(np.unique(freqlistm)))))
plt.ylabel('Frequencies (Hz)')
plt.title('Equal Tempered Intervals')

###  plotting melodicharmonic pitch occurences
plt.subplot(4, 2, 5)
plt.xlabel('Time')
plt.ylabel('Pitches')
plt.plot(melodicharmonic3,'bo')
plt.title('Melodic Successions & Harmonies')
plt.tight_layout()

#PLOT JI INTERVALS
```

Stellenbosch University https://scholar.sun.ac.za

```
plt.subplot(4, 2, 6)
plt.plot(list(freqlistr5),'ro')
plt.plot(list(freqs5),'bs')
plt.xlabel('%s Sample Pitches, %s Pitches found' %(
    (len(freqs5)-1),(len(freqlistr5))))
plt.ylabel('Found (Hz)')
plt.title('Rational Tuning Intervals')
plt.tight_layout()

#PLOT EQ INTERVALS
plt.subplot(4, 2, 7)
plt.plot(list(np.unique(freqlistless5)),'r^')
plt.plot(list(feqround5),'bo')
plt.xlabel('%s - TET Sample Pitches, %s Pitches found' %(
    (len(feqround5)-1),(len(np.unique(freqlistless5)))))
plt.ylabel('Frequencies (Hz)')
plt.title('Equally Tempered Intervals - 1st octave')
plt.tight_layout()

#PLOT EQ INTERVALS
plt.subplot(4, 2, 8)
plt.plot(list(np.unique(freqlistless51)),'r^')
plt.plot(list(feqround51),'bo')
plt.xlabel('%s - TET Sample Pitches, %s Pitches found' %(
    (len(feqround51)-1),(len(np.unique(freqlistless51)))))
plt.ylabel('Frequencies (Hz)')
plt.title('Equally Tempered Intervals - 2nd octave')
plt.tight_layout()

plt.show(3)

#### Analysis of Sample Fundamental Tones
# Open D
yOD, srOD, pitchesOD, magnitudesOD = TT.loadwavest(
    'Example D minor/16bit/Open D.wav')
cOD = TT.countpitch(pitchesOD)
domOD, highestOD, nexthighestOD, thirdhighestOD, fourthighestOD,
    fifthighestOD, sixthighestOD, seventhighestOD,
    eighthighestOD = TT.retestdomfreq(pitchesOD,
    magnitudesOD, cOD)
print 'Open D fundamental: %s' %domOD

# Stopped D
ySD, srSD, pitchesSD, magnitudesSD = TT.loadwavest(
```

Stellenbosch University https://scholar.sun.ac.za

```
    'Example D minor/16bit/Stopped D.wav')
cSD = TT.countpitch(pitchesSD)
domSD, highestSD, nexthighestSD, thirdhighestSD, fourthighestSD,
    fifthighestSD, sixthighestSD, seventhighestSD,
    eighthighestSD = TT.retestdomfreq(pitchesSD,
    magnitudesSD, cSD)
print 'Stopped D fundamental: %s' %domSD

# Stopped D with Vibrato
yVD, srVD, pitchesVD, magnitudesVD = TT.loadwavest(
    'Example D minor/16bit/Stopped D with vibrato.wav')
cVD = TT.countpitch(pitchesVD)
domVD, highestVD, nexthighestVD, thirdhighestVD, fourthighestVD,
    fifthighestVD, sixthighestVD, seventhighestVD,
    eighthighestVD = TT.retestdomfreq(pitchesVD,
    magnitudesVD, cVD)
print 'Vibrato D fundamental: %s' %domVD
```

B.1.3 Log Excerpts

```
SAMPLE 1 - gamut analysis

fundamental: 146.5
dominant: 146.1

Rational Intervals Found:
Scale step number 0 at 146.5 Hz with a ratio of 1:1
Scale step number 4 at 164.8 Hz with a ratio of 9:8
Scale step number 5 at 173.6 Hz with a ratio of 32:27
Scale step number 6 at 176.0 Hz with a ratio of 19683:16384
Scale step number 7 at 182.9 Hz with a ratio of 8192:6561
Scale step number 9 at 195.3 Hz with a ratio of 4:3
Scale step number 23 at 289.1 Hz with a ratio of 1048576:531441

7 Ratios from BregmanTK:
[1:1, 9:8, 32:27, 19683:16384, 8192:6561, 4:3, 1048576:531441]

0 known triads:
0 known tetrads:

1 known tetrachord:
(1:1, 9:8, 32:27, 4:3): 'PY-MinT'

0 known scales:
```

Stellenbosch University https://scholar.sun.ac.za

```
Tempered Intervals Found:
Scale step number 0 at 146.8 Hz
Scale step number 4 at 164.8 Hz
Scale step number 6 at 174.6 Hz
Scale step number 8 at 185.0 Hz
Scale step number 10 at 196.0 Hz
Scale step number 11 at 201.7 Hz
Scale step number 16 at 233.0 Hz
Scale step number 22 at 277.1 Hz

SAMPLE 1- melodic analysis

Successive Rational Intervals Found:
Scale step number 28 at 164.8 Hz with a ratio of 9:8
Scale step number 4 at 82.4 Hz with a ratio of 9:16
Scale step number 4 at 82.4 Hz with a ratio of 9:16
Scale step number 5 at 86.8 Hz with a ratio of 16:27
Scale step number 4 at 82.4 Hz with a ratio of 9:16
Scale step number 4 at 82.4 Hz with a ratio of 9:16
Scale step number 4 at 82.4 Hz with a ratio of 9:16
Scale step number 4 at 82.4 Hz with a ratio of 9:16

8 Ratios from BregmanTK:
[9:8, 9:16, 9:16, 16:27, 9:16, 9:16, 9:16, 9:16]

0 known triads:
0 known tetrads:
0 known tetrachords:
0 known scales:

Tempered Intervals Found:
Scale step number 4 at 164.8 Hz
Scale step number 6 at 174.6 Hz
Scale step number 16 at 233.0 Hz
Scale step number 24 at 293.6 Hz
Scale step number 6 at 349.2 Hz

SAMPLE 2 - gamut analysis

fundamental: 147.4
dominant: 177.3

Rational Intervals Found:
```

Stellenbosch University https://scholar.sun.ac.za

```
Scale step number 0 at 147.4 Hz with a ratio of 1:1
Scale step number 1 at 155.3 Hz with a ratio of 256:243
Scale step number 3 at 163.6 Hz with a ratio of 65536:59049
Scale step number 4 at 165.8 Hz with a ratio of 9:8
Scale step number 6 at 177.1 Hz with a ratio of 19683:16384
Scale step number 9 at 196.5 Hz with a ratio of 4:3
Scale step number 10 at 199.2 Hz with a ratio of 177147:131072
Scale step number 11 at 207.0 Hz with a ratio of 1024:729
Scale step number 12 at 209.9 Hz with a ratio of 729:512
Scale step number 14 at 221.1 Hz with a ratio of 3:2
Scale step number 15 at 232.9 Hz with a ratio of 128:81
Scale step number 16 at 236.1 Hz with a ratio of 6561:4096
Scale step number 22 at 279.8 Hz with a ratio of 243:128
Scale step number 23 at 290.8 Hz with a ratio of 1048576:531441

14 Ratios from BregmanTK:
[1:1, 256:243, 65536:59049, 9:8, 19683:16384, 4:3, 177147:131072,
1024:729, 729:512, 3:2, 128:81, 6561:4096, 243:128, 1048576:531441]

10 known triads:
(256:243, 4:3, 128:81): 'PY-maj',
(9:8, 3:2, 243:128): 'PY-maj',
(9:8, 6561:4096, 243:128): 'PY-dimtriad',
(177147:131072, 6561:4096, 243:128): 'PY-dimtriad',
(19683:16384, 729:512, 243:128): 'PY-maj',
(19683:16384, 6561:4096, 243:128): 'PY-min',
(9:8, 729:512, 243:128): 'PY-min',
(1:1, 4:3, 128:81): 'PY-min',
(9:8, 4:3, 243:128): 'PY-dimtriad',
(9:8, 4:3, 128:81): 'PY-dimtriad'

6 known tetrads:
(1:1, 9:8, 4:3, 128:81): 'PY-min6',
(19683:16384, 729:512, 6561:4096, 243:128): 'PY-maj6',
(9:8, 729:512, 6561:4096, 243:128): 'PY-min6',
(1:1, 256:243, 4:3, 128:81): 'PY-maj7',
(9:8, 729:512, 3:2, 243:128): 'PY-maj7',
(19683:16384, 177147:131072, 6561:4096, 243:128): 'PY-min6'

1 known tetrachord:
(19683:16384, 177147:131072, 729:512, 6561:4096): 'PY-MinT'
0 known scales:

Tempered Intervals Found:
```

Stellenbosch University https://scholar.sun.ac.za

```
Scale step number 0 at 146.8 Hz
Scale step number 4 at 164.8 Hz
Scale step number 6 at 174.6 Hz
Scale step number 10 at 196.0 Hz
Scale step number 11 at 201.7 Hz

SAMPLE 2 - melodic analysis

Rational Intervals Found:
Scale step number 0 at 73.7 Hz with a ratio of 1:2
Scale step number 24 at 147.4 Hz with a ratio of 1:1
Scale step number 24 at 147.4 Hz with a ratio of 1:1
Scale step number 30 at 177.1 Hz with a ratio of 19683:16384
Scale step number 30 at 177.1 Hz with a ratio of 19683:16384
Scale step number 38 at 221.1 Hz with a ratio of 3:2
Scale step number 38 at 221.1 Hz with a ratio of 3:2
Scale step number 38 at 221.1 Hz with a ratio of 3:2
Scale step number 48 at 294.8 Hz with a ratio of 2:1
Scale step number 0 at 73.7 Hz with a ratio of 1:2
Scale step number 46 at 279.8 Hz with a ratio of 243:128
Scale step number 4 at 82.9 Hz with a ratio of 9:16
Scale step number 4 at 82.9 Hz with a ratio of 9:16
Scale step number 3 at 81.8 Hz with a ratio of 32768:59049
Scale step number 3 at 81.8 Hz with a ratio of 32768:59049
Scale step number 39 at 232.9 Hz with a ratio of 128:81
Scale step number 4 at 82.9 Hz with a ratio of 9:16
Scale step number 3 at 81.8 Hz with a ratio of 32768:59049
Scale step number 34 at 199.2 Hz with a ratio of 177147:131072
Scale step number 40 at 236.1 Hz with a ratio of 6561:4096
Scale step number 4 at 82.9 Hz with a ratio of 9:16
Scale step number 4 at 82.9 Hz with a ratio of 9:16
Scale step number 4 at 82.9 Hz with a ratio of 9:16
Scale step number 4 at 82.9 Hz with a ratio of 9:16
Scale step number 30 at 177.1 Hz with a ratio of 19683:16384
Scale step number 34 at 199.2 Hz with a ratio of 177147:131072
Scale step number 24 at 147.4 Hz with a ratio of 1:1

27 Ratios from BregmanTK:
[1:2, 1:1, 1:1, 19683:16384, 19683:16384, 3:2, 3:2, 3:2, 2:1,
1:2, 243:128, 9:16, 9:16, 32768:59049, 32768:59049, 128:81,
9:16, 32768:59049, 177147:131072, 6561:4096, 9:16, 9:16,
9:16, 9:16, 19683:16384, 177147:131072, 1:1]

4 known triads:
```

Stellenbosch University https://scholar.sun.ac.za

```
(177147:131072, 6561:4096, 243:128): 'PY-dimtriad',
(9:16, 6561:4096, 243:128): 'PY-dimtriad',
(19683:16384, 6561:4096, 243:128): 'PY-min',
(9:16, 3:2, 243:128): 'PY-maj'

1 known tetrad:
(19683:16384, 177147:131072, 6561:4096, 243:128): 'PY-min6'

0 known tetrachords:
0 known scales:

Tempered Intervals Found:
Scale step number 0 at 146.8 Hz
Scale step number 6 at 174.6 Hz
Scale step number 24 at 293.6 Hz
Scale step number 6 at 349.2 Hz

SAMPLE 3 - gamut analysis

fundamental: 146.5
dominant: 220.2

Rational Intervals Found:
Scale step number 0 at 146.5 Hz with a ratio of 1:1
Scale step number 5 at 173.6 Hz with a ratio of 32:27

2 Ratios from BregmanTK:
[1:1, 32:27]
0 known triads:
0 known tetrads:
0 known tetrachords:
0 known scales:

Tempered Intervals Found:
Scale step number 3 at 160.1 Hz
Scale step number 4 at 164.8 Hz
Scale step number 10 at 196.0 Hz
Scale step number 14 at 220.0 Hz
Scale step number 16 at 233.0 Hz
Scale step number 22 at 277.1 Hz

SAMPLE 3 - melodic analysis

Rational Intervals Found:
```

Stellenbosch University https://scholar.sun.ac.za

```
Scale step number 24 at 146.5 Hz with a ratio of 1:1
Scale step number 4 at 82.4 Hz with a ratio of 9:16
Scale step number 4 at 82.4 Hz with a ratio of 9:16
Scale step number 4 at 82.4 Hz with a ratio of 9:16
Scale step number 29 at 173.6 Hz with a ratio of 32:27
Scale step number 48 at 293.0 Hz with a ratio of 2:1
Scale step number 48 at 293.0 Hz with a ratio of 2:1

7 Ratios from BregmanTK:
[1:1, 9:16, 9:16, 9:16, 32:27, 2:1, 2:1]
0 known triads:
0 known tetrads:
0 known tetrachords:
0 known scales:

Tempered Intervals Found:
Scale step number 3 at 160.1 Hz
Scale step number 4 at 164.8 Hz
Scale step number 10 at 196.0 Hz
Scale step number 14 at 220.0 Hz
Scale step number 22 at 277.1 Hz
Scale step number 3 at 320.2 Hz
Scale step number 10 at 391.9 Hz
Scale step number 14 at 439.9 Hz
Scale step number 22 at 554.2 Hz

Open D fundamental: 147.5
Stopped D fundamental: 148.0
Vibrato D fundamental: 147.4
```

B.1.4 Graph Output

Figure B.3 shows the results of the analysis of the first sample, that of the student cellist.

Figure B.4 shows the results of the analysis of the second sample, that of the professional cellist.

Figure B.5 shows the results of the analysis of the third sample, that of the sequenced MIDI file.

Stellenbosch University https://scholar.sun.ac.za

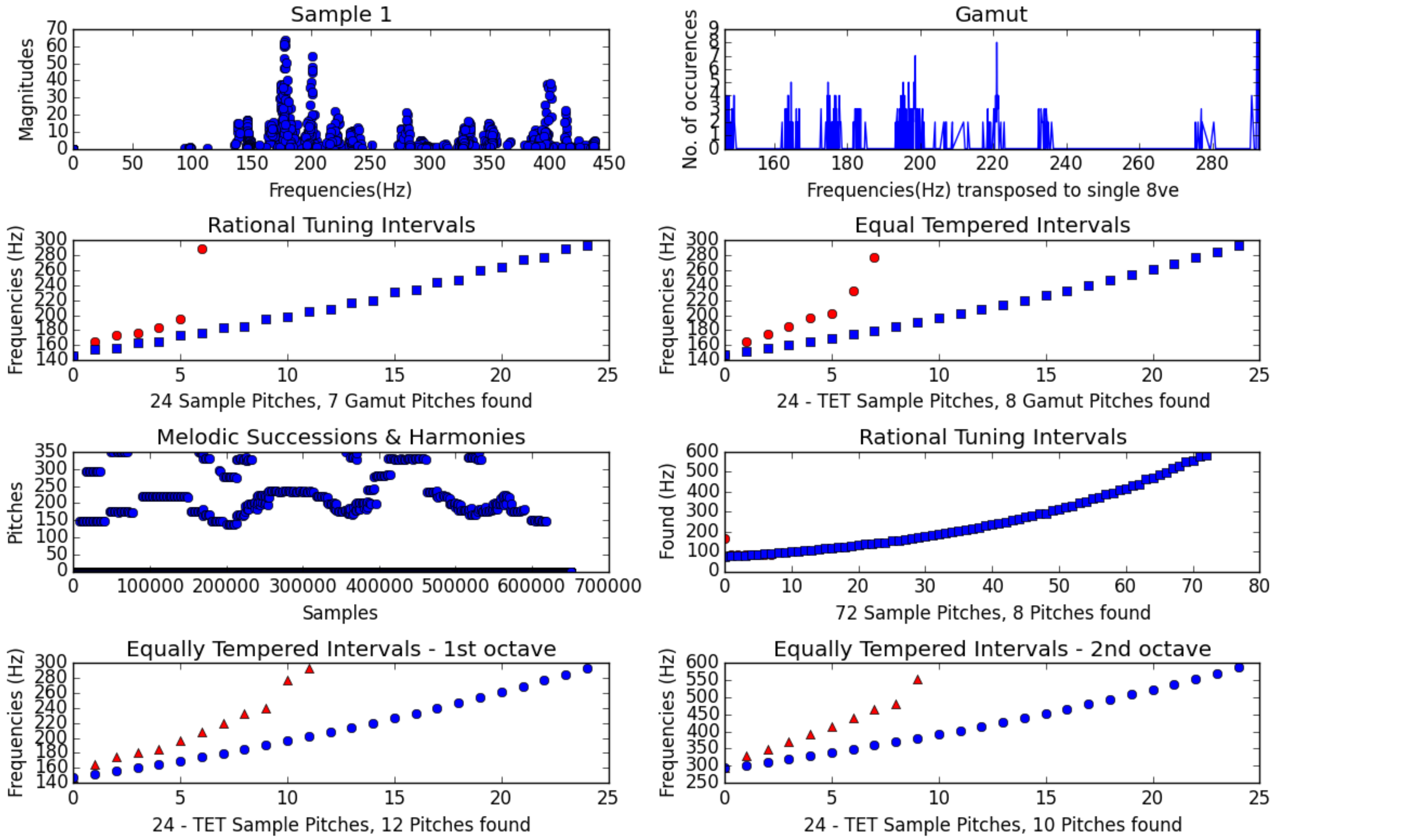

Figure B.3: Plots from Sample 1

Stellenbosch University https://scholar.sun.ac.za

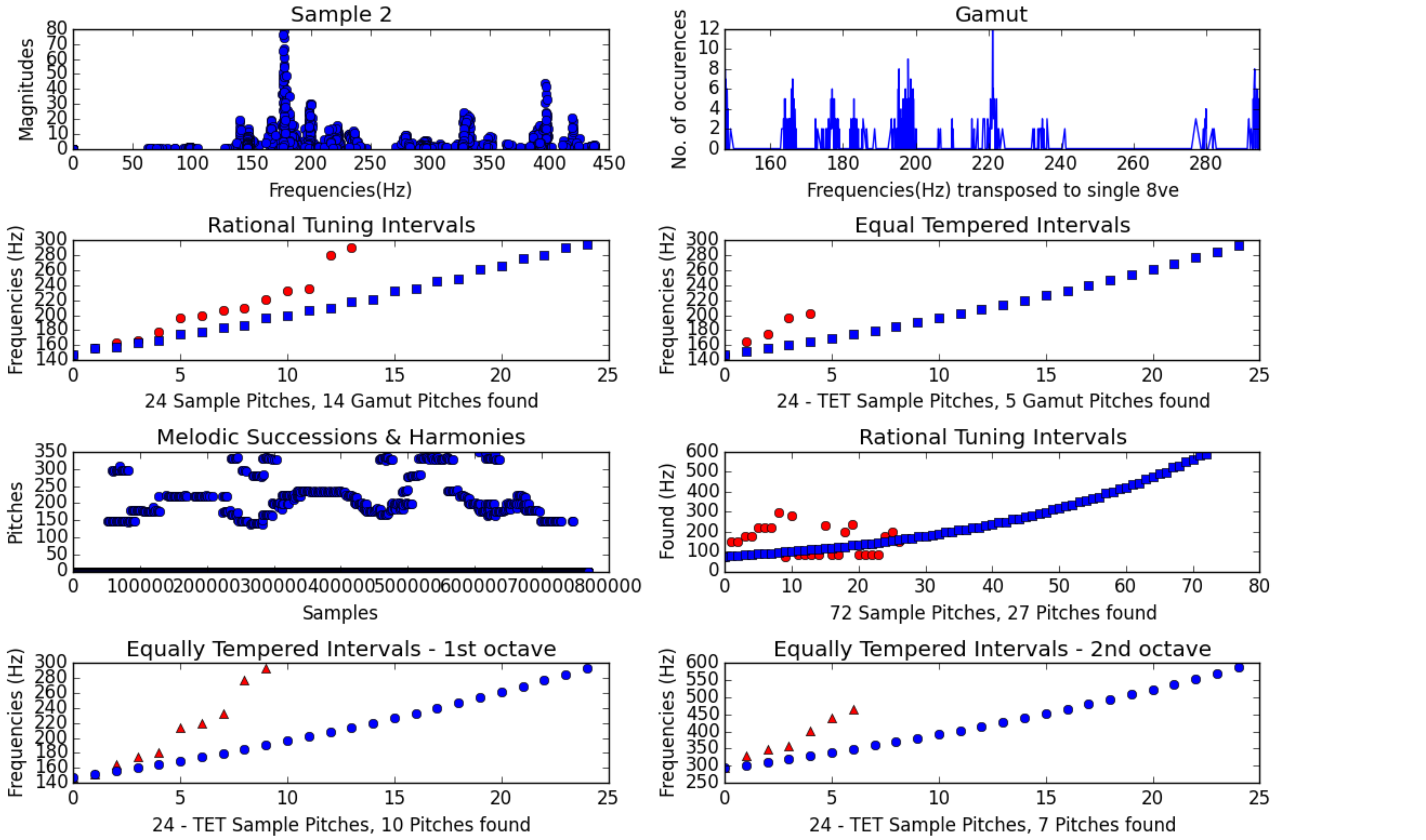

Figure B.4: Plots from Sample 2

Stellenbosch University https://scholar.sun.ac.za

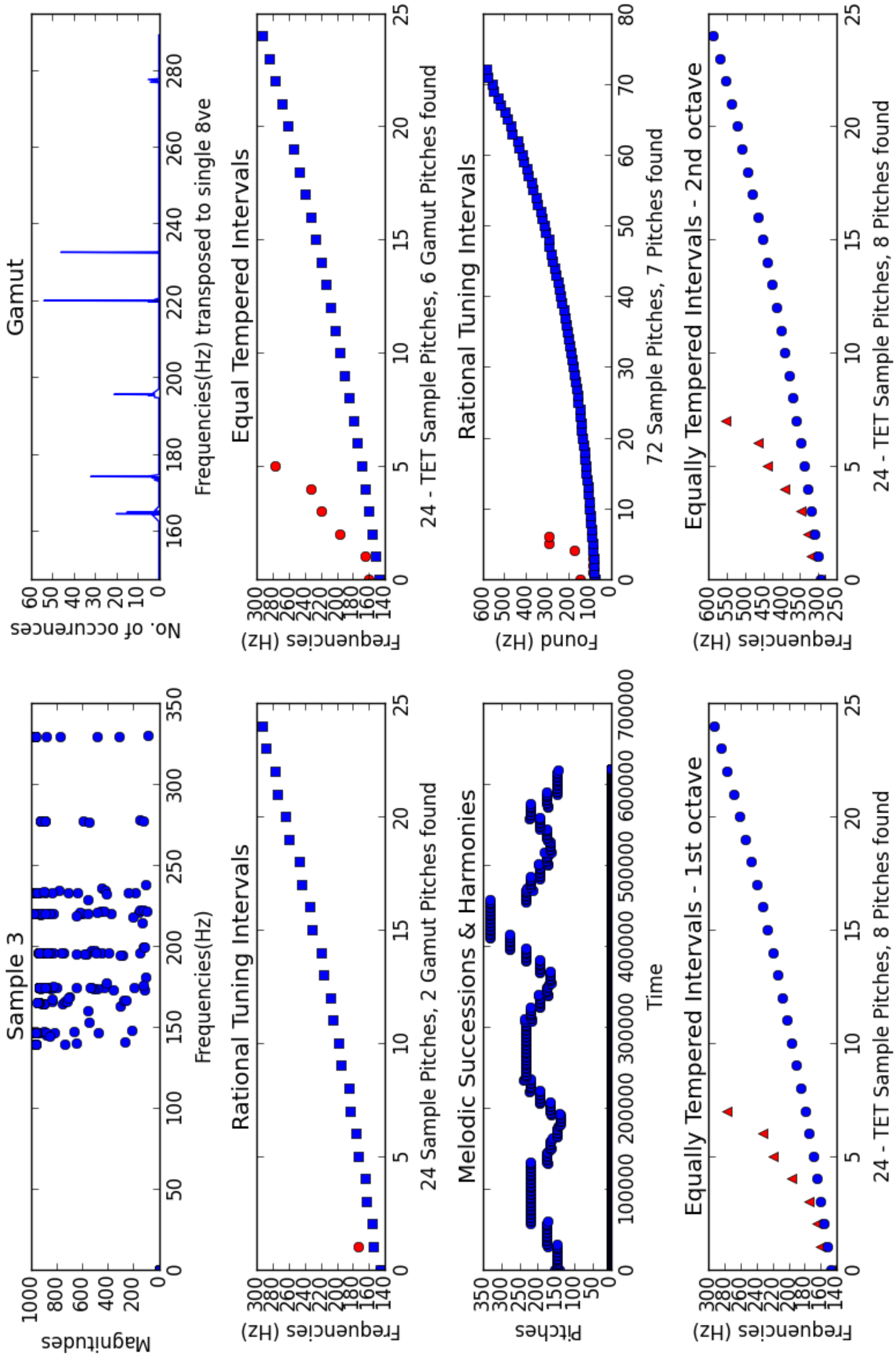

Figure B.5: Plots from Sample 3

Stellenbosch University https://scholar.sun.ac.za

List of References

Adkins, C. 1963. *The Theory and Practice of the Monochord.* State University of Iowa. 5

Al-Ghawanmeh, F.M., Al-Ghawanmeh, M.T. *&* Haddad, R.N. 2014. Proposing a process for using analysis software to improve authentic arab singing and ornamenting. volume 4, pages 47–56. Jordan: University of Jordan. 71

Albanese, D., Visintainer, R., Merler, S., Riccadonna, S., Jurman, G. *&* Furlanello, C. 2012. mlpy: Machine learning python. 64

Allauzen, C., Crochemore, M. *&* Raffinot, M. 1999. Factor oracle: A new structure for pattern matching. In *SOFSEM'99: Theory and Practice of Informatics*, pages 295–310. Springer. 63

Alpaydin, E. 2014. *Introduction to Machine Learning.* MIT press. 64

Apel, W. 1983. *Harvard Dictionary of Music.* 2nd edition. London: Heinemann. xiii

Aristoxenus *&* Macran, H.S. 1902. *Aristoxenou Harmonika Stoicheia: The Harmonics of Aristoxenus; Edited with Translation, Notes, Introduction and Index of Words.* Oxford: Clarendon Press. 3, 25

Association, I.M. *et al.*. 1983. Midi musical instrument digital interface specification 1.0. *Los Angeles.* 4

Backus, J. *&* Baskerville, R. 1977. *The Acoustical Foundations of Music*, volume 8. Norton New York. 3

Barker, A. 1989. *Greek Musical Writings, Volume II: Harmonic and Acoustic Theory.* Cambridge: Cambridge University Press. xv, xvi, 3, 9, 17, 20, 21, 22, 24, 25, 26, 27, 28, 33, 35, 38, 40, 41, 46, 47, 48, 49, 50, 99

Benson, D. 2007. *Music: A Mathematical Offering.* Cambridge University Press. xiv, 3, 62

Blackwood, E. 1991. Modes and chord progressions in equal tunings. *Perspectives of New Music*, pages 166–200. 18

Bogert, B.P., Healy, M.J. *&* Tukey, J.W. 1963. The quefrency analysis of time series for echoes: Cepstrum, pseudo-autocovariance, cross-cepstrum and saphe cracking.

Stellenbosch University https://scholar.sun.ac.za

In *Proceedings of the symposium on time series analysis*, volume 15, pages 209–243. chapter. 63

Bosanquet, R. 1874. The theory of the division of the octave, and the practical treatment of the musical systems thus obtained. revised version of a paper entitled'on just intonation in music; with a description of a new instrument for the easy control of systems of tuning other than the equal temperament of 12 divisions in the octave. *Proceedings of the Royal Society of London*, 23(156-163):390–408. xiv, 18, 31

Bosanquet, R. 1877. On the hindoo division of the octave, with some additions to the theory of systems of the higher orders. *Proceedings of the Royal Society of London*, 26(179-184):372–384. 3

Bradley, R.E., Edward Sandifer, C. *&* Williams, K. 2008. Leonhard euler: Life, work and legacy. *The Mathematical Intelligencer*, 30(2):66–69. 47

Brauchli, B., Butt, J. *&* Dreyfus, L. 1998. *The Clavichord.* Cambridge University Press Cambridge. 68

Brosbol, J. *&* Schubert, E. 2006. Calculating articulation in solo music performances. In *9th International Conference on Music Perception and Cognition*, pages 1362–1366. 75

Brower, C. 2008. Paradoxes of pitch space. *Music Analysis*, 27(1):51–106. 4

Busoni, F. 1911. *Sketch of a New Esthetic of Music.* New York: G. Schirmer. 4, 20

Cabrera, D., Ferguson, S. *&* Schubert, E. 2007. Psysound3: Software for acoustical and psychoacoustical analysis of sound recordings. In *Proc. 13th International Conference on Auditory Display*, pages 356–363. Montreal: ICAD. 74, 75

Carbonell, J.G., Michalski, R.S. *&* Mitchell, T.M. 1983. An overview of machine learning. In *Machine learning*, pages 3–23. Springer. 64

Carlos, W. 1987. Tuning: At the crossroads. *Computer Music Journal*, pages 29–43. 38, 52

Chalmers, J.H. 1993. *Divisions of the Tetrachord: A Prolegomenon to the Construction of Musical Scales.* Frog Peak Music. 27

Childers, D.G., Skinner, D.P. *&* Kemerait, R.C. 1977. The cepstrum: A guide to processing. *Proceedings of the IEEE*, 65(10):1428–1443. 63

Cho, G.J. 2003. *The Discovery of Musical Equal Temperament in China and Europe in the Sixteenth Century*, volume 93. Edwin Mellen Press. 3, 18

Cooley, J.W. *&* Tukey, J.W. 1965. An algorithm for the machine calculation of complex fourier series. *Mathematics of computation*, 19(90):297–301. 62

Creese, D. 2010. *The Monochord in Ancient Greek Harmonic Science.* Cambridge University Press. 5, 24

Stellenbosch University https://scholar.sun.ac.za

Dahlhaus, C. 1980. Tonality. *The New Grove Dictionary of Music and Musicians*, 19:51–55. 34

Dahlhaus, C. 1990. *Studies on the Origin of Harmonic Tonality.* Princeton, NJ: Princeton University Press. xiv

Daniélou, A. 1958. *Tableau Comparatif Des Intervalles Musicaux.* Institut Français d'Indologie. 12

Danielou, A. 1995. Music and the power of sound. *Inner Traditions: Rochester, VT.* 43, 50

De Groot, J. 2008. Dunamis and the science of mechanics: Aristotle on animal motion. *Journal of the History of Philosophy*, 46(1):43–67. 26

Dedekind, R. 1901. *Essays on the Theory of Numbers: I. Continuity and Irrational Numbers, II. The Nature and Meaning of Numbers.* Open court publishing Company. 18

Dubnov, S., Assayag, G. & Cont, A. 2011. Audio oracle analysis of musical information rate. In *Semantic Computing (ICSC), 2011 Fifth IEEE International Conference on*, pages 567–571. IEEE. 63

Dubnov, S., Assayag, G., Cont, A. *et al.*. 2007. Audio oracle: A new algorithm for fast learning of audio structures. In *Proceedings of International Computer Music Conference (ICMC)*. 63, 64

Duffin, R.W. 2008. *How Equal Temperament Ruined Harmony (and Why You Should Care).* WW Norton & Company. 4, 18, 20

Ellis, A.J. 1863*a*. On the conditions, extent, and realization of a perfect musical scale on instruments with fixed tones. *Proceedings of the Royal Society of London*, 13:93–108. 4, 20, 44, 47, 51, 52, 53, 73

Ellis, A.J. 1863*b*. On the temperament of musical instruments with fixed tones. *Proceedings of the Royal Society of London*, 13:404–422. xiii, 4, 20, 40, 51

Erlich, P. 1998. Tuning, tonality, and twenty-two-tone temperament. *Xenharmonikon*, 17:12–40. 3

Feng, Y., Zhuang, Y. & Pan, Y. 2003. Music information retrieval by detecting mood via computational media aesthetics. In *Web Intelligence, 2003. WI 2003. Proceedings. IEEE/WIC International Conference on*, pages 235–241. IEEE. 75

Glover, J.C., Lazzarini, V. & Timoney, J. 2011. Python for audio signal processing. 57

Gómez, E., Klapuri, A. & Meudic, B. 2003. Melody description and extraction in the context of music content processing. *Journal of New Music Research*, 32(1):23–40. 57, 60, 61, 62, 63, 66

Stellenbosch University https://scholar.sun.ac.za

Hawkins, J. 1776. *A General History of the Science and Practice of Music*, volume 4. London: T. Payne and Son. 11

Hawkins, J. 1853. *A General History of the Science and Practice of Music*, volume 2. Alfred Novello. 20, 69

Helmholtz, H. 1895. *On the Sensations of Tone*. 3rd edition. London: Longmans, Green and Co. xii, 1, 2, 3, 5, 6, 7, 8, 10, 13, 18, 19, 20, 21, 23, 24, 34, 38, 43, 47, 50, 53, 68, 94

Hiebert, E. 2014. The enharmonium. In *The Helmholtz Legacy in Physiological Acoustics*, chapter 12, pages 95–97. Springer. 68

Holdsworth, J., Nimmo-Smith, I., Patterson, R. & Rice, P. 1988. Implementing a gammatone filter bank. *Annex C of the SVOS Final Report: Part A: The Auditory Filterbank*, 1:1–5. 76

Hyer, B. 2001. Tonality. *The New Grove Dictionary of Music and Musicians*, 25:583–594. 33, 34

Hyer, B. 2002. Tonality. *The Cambridge History of Western Music Theory*, pages 726–752. 34

Itten, J. 1973. The art of color the subjective experience and objective rationale of color. 30

Kak, S. 2009. [Online] Early Record of Divisibility and Primality. Available at: `http://www.arxiv.org/pdf/0904.1154v1.pdf` [Accessed 2014-09-10]. 20, 22

Keegan, B.P., Tjoa, S.K. & Liu, K.R. 2011. Super-resolution of musical signals using approximate matching pursuit. In *Applications of Signal Processing to Audio and Acoustics (WASPAA), 2011 IEEE Workshop on*, pages 81–84. IEEE. 65

Keislar, D.F. 1988. *History and Principles of Microtonal Keyboard Design*. CCRMA, Department of Music, Stanford University. 17, 31, 69

Kim, Y.E., Chai, W., Garcia, R. & Vercoe, B. 2000. Analysis of a contour-based representation for melody. In *ISMIR*. 57

Klapuri, A. 2000. Qualitative and quantitative aspects in the design of periodicity estimation algorithms. In *Proceedings of the European Signal Processing Conference*. 63

Kohonen, T. 1998. The self-organizing map. *Neurocomputing*, 21(1):1–6. 75

Kohonen, T. 2001. *Self-Organizing Maps*, volume 30. Springer Science & Business Media. 75, 77

Kohonen, T., Hynninen, J., Kangas, J. & Laaksonen, J. 1995. Som pak, the self organizing map program package. *Lab. of Computer and Information Science, Helsinki University of Technology, Finland*, (A31). 77

Stellenbosch University https://scholar.sun.ac.za

Kornbluth, C.M. 1970. Gomez. *Argosy*, XXXI(9). 29

Lartillot, O. & Toiviainen, P. 2007. A matlab toolbox for musical feature extraction from audio. In *International Conference on Digital Audio Effects*, pages 237–244. Bordeaux, France. 75

Lerdahl, F. 1987. Timbral hierarchies. *Contemporary Music Review*, 2(1):135–160. 45

Lots, I.S. & Stone, L. 2008. Perception of musical consonance and dissonance: An outcome of neural synchronization. *Journal of The Royal Society Interface*, 5(29):1429–1434. xii, xiv

Loy, G. 1985. Musicians make a standard: The midi phenomenon. *Computer Music Journal*, pages 8–26. 4

Lyon, R. 1978. A signal-processing model of hearing. Technical Report, tech. report, Xerox PARC. 76

Lyon, R.F. 1982. A computational model of filtering, detection, and compression in the cochlea. In *Acoustics, Speech, and Signal Processing, IEEE International Conference on ICASSP'82.*, volume 7, pages 1282–1285. IEEE. 76

Lyon, R.F. 1998. Filter cascades as analogs of the cochlea. In *Neuromorphic systems engineering*, pages 3–18. Springer. 76

Lyon, R.F. 2010. Machine hearing: An emerging field [exploratory dsp]. *Signal Processing Magazine, IEEE*, 27(5):131–139. 64, 76

Lyon, R.F. & Mead, C. 1988. An analog electronic cochlea. *Acoustics, Speech and Signal Processing, IEEE Transactions on*, 36(7):1119–1134. 76

Maher, R.C. & Beauchamp, J.W. 1994. Fundamental frequency estimation of musical signals using a two-way mismatch procedure. *The Journal of the Acoustical Society of America*, 95(4):2254–2263. 63

Mandelbaum, J. 1961. Multiple division of the octave and the tonal resources of 19-tone temperament. Ph.D. thesis. 18, 50, 51

Mathews, M.V. & Pierce, J.R. 1989. The bohlen-pierce scale. In *Current directions in computer music research*, pages 165–173. MIT Press. 38, 52

Mathews, M.V., Pierce, J.R., Reeves, A. & Roberts, L.A. 1988. Theoretical and experimental explorations of the bohlen–pierce scale. *The journal of the Acoustical Society of America*, 84(4):1214–1222. 38

McClain, E.G. 1978. *The Myth of Invariance: The Origin of the Gods, Mathematics and Music from the Rg Veda to Plato*. Boston, Mass.: Shambala Publishing. xv, xvi

McLeod, P. & Wyvill, G. 2003. Visualization of musical pitch. In *Computer Graphics International Conference*, pages 300–300. IEEE Computer Society. 72

Stellenbosch University https://scholar.sun.ac.za

McLeod, P. & Wyvill, G. 2005. A smarter way to find pitch. In *Proceedings of International Computer Music Conference, ICMC*. 8, 9, 61, 62, 72, 73, 74

Medan, Y., Yair, E. & Chazan, D. 1991. Super resolution pitch determination of speech signals. *Signal Processing, IEEE Transactions on*, 39(1):40–48. 65

Meddis, R., Hewitt, M.J. & Shackleton, T.M. 1990. Implementation details of a computation model of the inner hair-cell auditory-nerve synapse. *The Journal of the Acoustical Society of America*, 87(4):1813–1816. 76

Milne, A., Sethares, W. & Plamondon, J. 2007. Isomorphic controllers and dynamic tuning: Invariant fingering over a tuning continuum. *Computer Music Journal*, 31(4):15–32. xiii, 16, 17, 29, 31, 32, 67

Milne, A., Sethares, W. & Plamondon, J. 2008. Tuning continua and keyboard layouts. *Journal of Mathematics and Music*, 2(1):1–19. 17, 68, 69

Milne, A.J. 2013. A computational model of the cognition of tonality. Ph.D. thesis, The Open University. xiv, 33, 34, 67

Mozart, L. 1985. *A Treatise on the Fundamental Principles of Violin Playing*. Oxford: Oxford University Press. 20

Nabney, I. 2002. *NETLAB: Algorithms for Pattern Recognition*. Springer Science & Business Media. 75

Newman, E.B. 1933. The validity of the just noticeable difference as a unit of psychological magnitude. *Transactions of the Kansas Academy of Science (1903)*, pages 172–175. 94

Newton, I. 1965. A letter of mr. isaac newton, professor of the mathematicks in the university of cambridge; containing his new theory about light and colors: Sent by the author to the publisher from cambridge, febr. 6. 1671/72; in order to be communicated to the r. society. *Philosophical Transactions (1665-1678)*, pages 3075–3087. 30

Oliphant, T.E. 2006. *A Guide to NumPy*, volume 1. Trelgol Publishing USA. 57

Oppenheim, A.V. & Schafer, R.W. 2004. From frequency to quefrency: A history of the cepstrum. *Signal Processing Magazine, IEEE*, 21(5):95–106. 63

Partch, H. 1949. *Genesis of a Music; with a Foreword by Otto Luening*. University of Wisconsin Press. 4

Pesic, P. 2007. *Beyond Geometry: Classic Papers from Riemann to Einstein*. Courier Dover Publications. 28, 29

Pesic, P. 2013. Helmholtz, riemann, and the sirens: Sound, color, and the "problem of space". *Physics in Perspective*, 15(3):256–294. 4, 28, 29, 30

Pierce, J.R. 1983. *The Science of Musical Sound*. Scientific American Library New York. 3

Stellenbosch University https://scholar.sun.ac.za

Politis, D., Linardis, P. & Mastorakis, N. 2002. The arity of delta prosodic and musical interfaces: A metric of complexity for 'vector' sounds. In *Proceedings, 2nd International Conference, Music and Artificial Intelligence: ICMAI*, volume 2, pages 12–14. ICMAI, Edinburgh, Scotland: ICMAI2002. 30

Politis, D. & Margounakis, D. 2003. In search for chroma in music. In *Proceedings of the 7th WSEAS International Conference on Systems*. Corfu: WSEAS. 30

Prechtl, A., Milne, A.J., Holland, S., Laney, R. & Sharp, D.B. 2012. A midi sequencer that widens access to the compositional possibilities of novel tunings. *Computer Music Journal*, 36(1):42–54. 68, 69, 70

Prout, E. 1903. *Harmony: Its Theory and Practice*. 9182, 16th edition. London: Augener. 33

Rao, V. & Rao, P. 2008. Melody extraction using harmonic matching. *MIREX Audio Melody Extraction Contest Abstracts*. 63

Rodet, X., Potard, Y. & Barriere, J.-B. 1984. The CHANT project: From the synthesis of the singing voice to synthesis in general. *Computer Music Journal*, pages 15–31. 45

Ruckmick, C. 1929. A new classification of tonal qualities. *Psychological Review*, 36(2):172. 42

Scherchen, H. 1950. The nature of music, trans. *William Mann, London*. 6

Seneff, S. 1986. A computational model for the peripheral auditory system: Application of speech recognition research. In *Acoustics, Speech, and Signal Processing, IEEE International Conference on ICASSP'86.*, volume 11, pages 1983–1986. IEEE. 76

Sethares, W.A. 2005. *Tuning, Timbre, Spectrum, Scale*, volume 2. Springer. 45

Sethares, W.A., Milne, A.J., Tiedje, S., Prechtl, A. & Plamondon, J. 2009. Spectral tools for dynamic tonality and audio morphing. *Computer Music Journal*, 33(2):71–84. 67

Shepard, R.N. 1982. Geometrical approximations to the structure of musical pitch. *Psychological Review*, 89(4):305. 36, 37, 39, 41, 42, 74

Slaney, M. 1998. Auditory toolbox. *Interval Research Corporation, Tech. Rep*, 10:1998. 75, 76

Smith, G. 2006. Xenharmony. *Available online at http://lumma.org/tuning/gws/home.htm (Accessed October 2015.)*. 31

Sor, F. & Harrison, F.M. 1924. *Method for the Guitar*. Condensed, rewritten and edited edition. London: Robert Cooks and Co. 11

Spanias, A., Painter, T. & Atti, V. 2006. *Audio Signal Processing and Coding*. John Wiley & Sons. 65

Stellenbosch University https://scholar.sun.ac.za

Stoica, P. & Moses, R.L. 2005. *Spectral Analysis of Signals*. Pearson/Prentice Hall Upper Saddle River, NJ. 65

Sumner, C.J., Lopez-Poveda, E.A., O'Mard, L.P. & Meddis, R. 2002. A revised model of the inner-hair cell and auditory-nerve complex. *The Journal of the Acoustical Society of America*, 111(5):2178–2188. 76

Telgarsky, R. 2013. [Online] Dominant Frequency Extraction. Available at: `http://arxiv.org/pdf/1306.0103.pdf` [Accessed 2014-09-10]. 65

Terhardt, E., Stoll, G. & Seewann, M. 1982. Algorithm for extraction of pitch and pitch salience from complex tonal signals. *The Journal of the Acoustical Society of America*, 71(3):679–688. 74

Thwaites, S. & Fletcher, N.H. 1981. Some notes on the clavichord. *The Journal of the Acoustical Society of America*, 69(5):1476–1483. 68

Torrence, C. & Compo, G.P. 1998. A practical guide to wavelet analysis. *Bulletin of the American Meteorological society*, 79(1):61–78. 66, 67

Tovey, D.F. 1973. *Beethoven*, volume 101. Oxford University Press. 95

Tymoczko, D. 2012. The generalized tonnetz. *Journal of Music Theory*, 56(1):1–52. 68

Vesanto, J. 1999. SOM-based data visualization methods. *Intelligent data analysis*, 3(2):111–126. 77

Vesanto, J. 2000. Neural network tool for data mining: SOM toolbox. In *Proceedings of symposium on tool environments and development methods for intelligent systems (TOOLMET2000)*, pages 184–196. 77

Vesanto, J. & Alhoniemi, E. 2000. Clustering of the self-organizing map. *Neural Networks, IEEE Transactions on*, 11(3):586–600. 77, 78

Vesanto, J., Himberg, J., Alhoniemi, E. & Parhankangas, J. 1999. Self-organizing map in matlab: The SOM toolbox. In *Proceedings of the Matlab DSP conference*, volume 99, pages 16–17. 77

Vesanto, J., Himberg, J., Alhoniemi, E. & Parhankangas, J. 2000. *SOM Toolbox for Matlab 5*. Citeseer. 77

Vogel, S. 1993. Sensation of tone, perception of sound, and empiricism: Helmholtz's physiological acoustics. *Cahan*, 1993:259–87. 30

Von Goethe, J.W. & Eastlake, C.L. 1970. *Theory of Colours*. 3. Mit Press. 30

Wason, R.W., Marvin, E.W. & Riemann, H. 1992. Riemann's "ideen zu einer'lehre von den tonvorstellungen": An annotated translation. *Journal of Music Theory*, pages 69–79. 21

West, M.L. 1992. *Ancient Greek Music*. Oxford University Press. 3, 20, 30